Mind the Gap

Hyperfine Structure Theory and Morasses

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Abstract

Using the Friedman-Koepke Hyperfine Structure Theory, we construct first a gap-1-morass and then a gap-2-morass with perfect preservation.

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0 Preliminaries

Introduction

The constructible universe L of set theory is defined as the class of sets definable in a transfinite process as follows: We start with an empty L_0, for L_α already defined let $L_{\alpha+1}$ consist of all subsets of L_α definable by $\in$-formulae, and for limit ordinals λ take the union of all previous stages of the construction, $L_\lambda = \bigcup\limits_{\alpha<\lambda} L_\alpha$. Finally $L = \bigcup\limits_{\alpha\in\mathrm{On}} L_\alpha$.

L turns out to be the smallest inner model of set theory, i.e., it is a subclass of every other inner model. As a consequence of its very concrete definition, L has some fundamental properties undecidable in ZFC alone. Actually, Gödel defined this model to prove the consistency of ZFC with the continuum hypothesis (CH). This proof is based on the *condensation lemma* which states that Σ_1-substructures of L condense down to stages of L.

In contrast to the simplicity of its definition, the proofs of L's important properties such as the $\Box$-principle or the covering lemma can become rather complex. Ronald Jensen [7] in 1972 established those results, using his so-called *fine structure theory*. Even today, after 30 years of development, this method remains difficult.

In the early seventies, Jack Silver found a different approach — the Silver machines (see Richardson [11]). These machines reduce the considerations to calculations with sets of ordinals. Similarly to L, a new hierarchy for these sets, M^δ, is defined. Analogously to the condensation lemma we have a *collapsing property*, i.e., closed structures (which are produced by a hull operator) condense down to stages of the machine. In contrast to the constructible universe very little happens in the passage from M^δ to $M^{\delta+1}$. This is guaranteed by a certain *finiteness property* which codes all information needed for this step in a finite set which itself has a simple form. A development of this can be found in my Diplom thesis [10].

Another approach is due to Friedman and Koepke [4]; it incorporates the finiteness property and other ideas of Silver machines into the L hierarchy. In this thesis we use this *hyperfine structure theory* to build morasses. *Morasses* are combinatorial structures invented by Jensen as a tool to construct infinite structures from structures of smaller cardinality, e.g., a structure of

size $\aleph_3$ from countable structures uses a gap-2-morass. Important applications are the gap-$(n+1)$-transfer theorems (requiring gap-n-morasses). For a development of these ideas see Devlin [1].

The definition of a *gap-1-morass* (definition 10) is standard. Richardson [11] has a construction using Silver machines. Koepke [8] offered an outline of a gap-1-morass construction using hyperfine structure in a seminar held in Bonn. In this thesis the latter proof is completed, in particular by introducing a precise language for hyperfine structures, and simplified by what we call *type preservation* (lemma 8); this basically says that isomorphic hulls are still isomorphic when mapped in a Σ_1-preserving way; using type preservation one can avoid lengthy calculations using terms in the language of hyperfine structure theory.

The definition of a *gap-2-morass* (definition 15) is very natural: First of all the gap-2-morass shall also have the properties of the gap-1-morass, in addition "gap-2-maps" (mapping $\aleph_1$-like points with $\aleph_2$-like points on top) shall preserve "gap-1-maps". We achieve what we call "*perfect preservation*" meaning that gap-2-maps preserve all the morass-points and their relations (level and morass predecessors) below. In the context of Velleman's [13] simplified gap-2-morass, the expression "perfect tree preservation" is used. Our emphasis is on optimal preservation and continuity properties and not on how morasses are used in applications.

As an example of a property of our morass not resulting from Jensen's construction [6] (see Friedman [3] for a construction in the flavor of Jensen's construction using his Σ^*-fine-structure), our gap-2-maps preserve morass predecessors of $\aleph_3$-like morass-points while in Jensen's construction in general only their levels are preserved (see Morgan [9]). See remark 31 for more details.

The constructions of the gap-1 and the gap-2 morasses are very different in nature. Whilst the gap-1 construction (or the classical gap-2-morass approach) relies on maps with sufficient elementarity, our gap-2 construction takes the gap-1-maps and directly imposes further restrictions which force most of the new properties to be true; the burden of the gap-2 proof is then to again show that we still have a collection of maps with the gap-1-morass properties.

To consider higher-gap-morasses a similar notion of perfect preservation would require the new morass maps to preserve all morass maps of lower levels as well as the structure of morass-points below. Restrictions to the morass maps in the spirit of definition 17 would be needed to guarantee this.

Notation

The basic concepts of set theory (especially the constructible universe L) are assumed to be known. Any notation and definition not explained is standard and may, e.g., be found in Drake [2].

We use the usual logical symbols: $\wedge$ (and), $\vee$ (or), $\neg$ (not), $\exists$ (exists), $\forall$ (for all), $\rightarrow$ (implies), (, and) (parentheses)

For two sets x and y we write $x \cong y$ if x and y are isomorphic (i.e., there exists a 1-1 function from x onto y which preserves all structures on x; the structures will be clear from the context). Furthermore, we write $x \subset y$ if x is a (not necessarily proper) subset of y. For a well-ordering $\langle Z, <_Z \rangle$ and a set $X \subset Z$ let $\operatorname{lub} X$ (least upper bound) be the $<_Z$-least $z \in Z$ s. t. $\forall x \in X \; x < z$. As usual small Greek letters will denote ordinals.

Let $f \colon x \rightarrow y$; we write $\operatorname{dom} f$ for the domain and $\operatorname{range} f$ for the range of f. ${}^{<\omega}x$ is the set of all finite sequences in x. If x and y are ordered sets and f is a function which preserves this order we write $f \colon x \xrightarrow{\text{o. p.}} y$.

The Friedman-Koepke Hyperfine Structure Theory

Let's recall the basic definitions and properties. See [4] for details and proofs. The main tools of the theory are *locations*, also referred to as *names*, and the corresponding *hulls*. Locations are triples of the form $(\alpha, \varphi, \vec{x})$ well-ordered by $\tilde{\leq}$ (such a location will be called α-*location*, we will also refer to it as the level of the location). For a given location s we write $s = (\alpha(s), \varphi_{n(s)}, \vec{x}(s))$. The basic operations are:

Interpretation $I(\alpha, \varphi, \vec{x}) = \{y \in L_\alpha \mid L_\alpha \models \varphi(y, \vec{x})\}$

Naming For $y \in L$ let $N(y) = (\alpha, \varphi, \vec{x})$ be $\tilde{\leq}$-least s. t. $I(\alpha, \varphi, \vec{x}) = y$.

Skolem function $S(\alpha, \varphi, \vec{x})$ is the least $y \in L_\alpha$ s. t. $L_\alpha \models \varphi(y, \vec{x})$ if exists; else $S(\alpha, \varphi, \vec{x})\uparrow$ (undefined).

We say that $(\alpha, \varphi, \vec{x}) \in X \subset L$ if α and each component of $\vec{x}$ are elements of X. A set or class $X \subset L$ is *constructibly closed* iff X is closed under applications of I, N, and S. We denote the constructible *closure* or *hull* of X by $L\{X\}$. If X is constructibly closed and $\pi\colon X \cong M$ is the Mostowski collapse, then $M = L_\alpha$ for some $\alpha \in \mathrm{On}$ and the basic operations are preserved.

The *fine constructible hierarchy* is given by

$$L_s = \left(L_{\alpha(s)}, \in, <_L, I, N, S{\restriction}s\right)$$

where $S{\restriction}s$ means that S is applied to locations in $L_{\alpha(s)}$ and to $\alpha(s)$-locations below s (the latter will for that purpose also be considered elements of the structure, but not of the domain of I). Now the definition of closure extends to structures L_s for a location s, namely a set $X \subset L_{\alpha(s)}$ is closed in L_s ($X \lhd L_s$) if it is closed under its operations (S can be applied to top-locations below s if their third component is an element of X). The hull $L_s\{X\}$ is defined similarly. Again, we have *condensation*: There is a unique isomorphism $\pi\colon L_s\{X\} \cong L_{\bar{s}}$ for some $\bar{s}$. Locations are mapped component-wise; if the first component is $\alpha(s)$ it is mapped to $\alpha(\bar{s})$. For notational convenience we write $\pi(s) = \bar{s}$.

For finite sets $p, q \subset L_{\alpha(s)}$ define $p <^* q$ iff $\max_{<_L}(p \triangle q) \in q$ ($\triangle$ is the symmetric difference). If a finite set is used as a parameter to a formula, it is taken as a $<_L$-increasing tuple.

Additionally, we have a *finiteness property*, *monotonicity*, *continuity*, and a *compactness property*:

Finiteness Property For an α-location s there exists $z \in L_\alpha$ s. t. for any $X \subset L_\alpha$ we have $L_{s^+}\{X\} \subset L_s\{X \cup \{z\}\}$ where s^+ denotes the immediate successor of s in the well-ordering of locations; $z = S(s)$ is as required.

Monotonicity For α-locations $s \mathrel{\widetilde{\leq}} t$: $\forall X \subset L_\alpha\ L_s\{X\} \subset L_t\{X\}$

For s, t α-, β-locations respectively, where $\alpha < \beta$:

$$L_s\{X\} \subset L_t\{X \cup \{\alpha\}\}$$

Continuity For locations of the form $s = (\alpha, \varphi_0, \emptyset)$ for $\lim \alpha$ and $X \subset L_\alpha$:

$$L_s\{X\} = L\{X\} = \bigcup_{\beta<\alpha} L_{(\beta,\varphi_0,\emptyset)}\{X \cap L_\beta\}$$

For $s = (\alpha+1, \varphi_0, \emptyset)$ and $X \subset L_\alpha$:

$$\begin{aligned} & L_s\{X \cup \{\alpha\}\} \cap L_\alpha = L\{X \cup \{\alpha\}\} \cap L_\alpha \\ & = \bigcup \{L_t\{X\} \mid t \text{ an } \alpha\text{-location}\} \end{aligned}$$

For $s = (\alpha, \varphi, \vec{x})$ a $\tilde{\leq}$-limit not of the above forms and $X \subset L_\alpha$:

$$L_s\{X\} = \bigcup \left\{ L_t\{X\} \,\middle|\, t \mathrel{\tilde{<}} s \text{ an } \alpha\text{-location} \right\}$$

Compactness Property Let s be an α-location and $X \subset L_\alpha$, then $x \in L_s\{X\}$ iff $x \in L_s\{Y\}$ for some finite $Y \subset X$.

Lemma 1 *Let s be a γ-location, $X \lhd L_s$, $\pi\colon X \cong L_{\bar{s}}$, and $t \mathrel{\tilde{\leq}} s$, $t \in X \cup \{\gamma\}$, then (let $\alpha = \alpha(t)$):*

$$\forall Z \subset X \cap L_\alpha \; \pi[L_t\{Z\}] = L_{\pi(t)}\{\pi[Z]\},$$

Proof First note, that $X \cap L_\alpha \lhd L_t$. Hence $\pi \restriction X \cap L_\alpha \colon X \cap L_\alpha \cong L_{\bar{t}}$ where $\bar{t} = \tilde{\leq}\text{-lub}\, \pi\left[\left\{ r \mathrel{\tilde{<}} t \,\middle|\, r \in X \cap L_\alpha \cup \{\alpha\} \right\}\right]$ (with $\pi(\gamma) = \alpha(\bar{s})$). Then, of course, $\pi[L_t\{Z\}] = L_{\bar{t}}\{\pi[Z]\}$ for $Z \subset X \cap L_\alpha$. It remains to show that $\bar{t} = \pi(t)$.

Since π preserves the $\tilde{\leq}$-relation, $\bar{t} \mathrel{\tilde{\leq}} \pi(t)$. On the other hand, let $r = (\beta, \varphi, \vec{b}) \mathrel{\tilde{<}} \pi(t)$. Then $\beta \leq \pi(\alpha)$ and $\vec{b} \in L_\beta \subset L_{\pi(\alpha)} = \pi[X \cap \alpha]$. So there are $\delta \in X \cap L_\alpha \cup \{\alpha\}$ and $\vec{d} \in X \cap L_\alpha$ s. t. $r = (\pi(\delta), \varphi, \pi(\vec{c}))$. But then $\pi^{-1}(r) = (\delta, \varphi, \vec{c}) \mathrel{\tilde{<}} t$ and $\pi^{-1}(r) \in X \cap L_\alpha \cup \{\alpha\}$. Therefore, by definition of $\bar{t}$ we have $r \mathrel{\tilde{<}} \bar{t}$. ⊣

Next we fix our language for the investigation of morasses.

Definition 2 (Language $\mathcal{L}$ for L_s) Let s be an α-location. We take *function symbols* for the structure L_s discussed above: naming N, interpretation I, Skolem function S, location decomposition $\alpha(\cdot)$ and $\vec{x}(\cdot)$, and location composition $(\cdot, \varphi., \cdot)$. (The distinction between function symbols and functions won't be shown, same for relations etc.). We have *relation symbols* $\in$, $<_L$ (on sets, i.e., on elements of the structure), $=$ (on sets and locations) and $\tilde{<}$, $\tilde{\leq}$ (on locations). Finally, we have *variables* for sets.

Terms are defined as usual, note that there will be terms for sets and for locations: Variables are terms. If x, y are set terms or y is α (strictly speaking a constant symbol for the top level) and t a location term, then the following are also terms: $N(x)$, $I(t)$, $S(t)$, $\alpha(t)$, $\vec{x}(t)$, (y, φ_n, x) for $n < \omega$.

Interpretation of terms. Given a term t with variables v_i, $i < k$ for some $k < \omega$, interpreted as $a_i \in L_\alpha$. Then the interpretation t^s of t is defined inductively: If t is of the form v_i then $t^s = a_i$. If t is of the form (t_0, φ_n, t_1) and t_0^s is defined and an ordinal or α, $n < \omega$, and t_1^s is defined and a vector of length m of elements of $L_{t_0^s}$ where φ_n has $m+1$ free variables ("t_1 is of the right length"), then $t^s = (t_0^s, \varphi_n, t_1^s)$ provided that this is $\widetilde{<}\ s$, else undefined. If $t^s = (\beta, \varphi_n, \vec{z})$ is defined with $\beta < \alpha$ then $\alpha(t)^s = \beta$ and $I(t)^s = I(t^s)$. If $t^s = (\beta, \varphi_n, \vec{z})$ is defined then $\vec{x}(t)^s = \vec{z}$ and $S(t)^s = S(t^s)$ (here $\beta \leq \alpha$). If t^s is defined and t a set term then $N(t)^s = N(t^s)$. All other terms are undefined, we write $t\!\uparrow$; also $t\!\downarrow$ iff $\neg\, t\!\uparrow$.

We say that *a term t is determined by location s* iff for each subterm of the form (α, φ_n, t) where t^s is defined, if (α, φ_n, t^s) is a location then it is $\widetilde{<}\ s$.

Given set terms x_0, x_1 as well as location terms t_0, t_1 the following are *atomic formulas*: $x_0 \in x_1$, $x_0 <_L x_1$, $x_0 = x_1$, $t_0 = t_1$, $t_0 \ \widetilde{<}\ t_1$ and $t_0 \ \widetilde{\leq}\ t_1$. Atomic formulas are *formulas*. And if φ, χ are formulas and v is a variable, then $\varphi \wedge \chi$, $\neg\, \varphi$ and $\exists v\, \varphi(v)$ are formulas. A *quantifier-free formula (QFF)* is a formula with no occurrence of $\exists$. A Σ_1*-formula* is a formula of the form $\exists v\, \varphi(v)$ where φ is quantifier-free; instead of v a tuple $\vec{v}$ is allowed.

We say that *a formula φ is determined by location s* iff each term in it is determined by location s.

Given an assignment of the variables, we define truth for a determined formula φ ($L_s \models \varphi$) as follows: Equality is true in L_s iff both sides are defined and equal or both sides are undefined. The other relations must have both sides defined to be true. $\varphi \wedge \chi$ is true iff φ and χ are true, $\neg\, \varphi$ is true iff φ is false and $\exists v\, \varphi(v)$ is true iff there is an $a \in L_\alpha$ s. t. $\varphi(a)$ holds.

The hull of X for the location s, $L_s\{X\}$, is the set of values of defined terms with parameters from X.

The Σ_1-*hull* of X for the location s is the closure of the normal hull $L_s\{X\}$ under $<_L$-least witnesses for Σ_1-formulas. We write $L_s^*\{X\}$.

Remark 3 *The following observations about our language are straightforward (t a term, φ a formula, given an assignment):*

Assume t is determined by s. Then so is every subterm of t. Further, if $s \ \widetilde{<}\ s'$ with $\alpha(s) = \alpha(s')$ then t is determined by s' and t^s is defined iff $t^{s'}$

is defined, in which case their values agree. If s is a limit location then t is already determined by a location $s' \mathrel{\tilde{<}} s$ (note that if s is a minimal location with $\alpha(s)$ a successor ordinal, it will be formally necessary to replace terms interpreted as $\alpha(s')$, if any, by the constant symbol for the top level of $L_{s'}$); furthermore, s' can be taken from $L_s\{\vec{a}\}$ where $\vec{a}$ is assigned to the free variables of t. The latter implies that a structure-preserving map between structures L_s with s limit preserves the determinedness of terms.

If t is determined by s, t^s is defined, $s \mathrel{\tilde{<}} s'$ with $\alpha(s) < \alpha(s')$, then t' is determined by s' with $t^s = (t')^{s'}$ where t' is the same as t with all references to the top level α replaced by $\alpha(s)$.

If t is determined by s, then $t\!\uparrow$ (and hence also $t\!\downarrow$) can be expressed by a QFF: If t is a set term we have $t\!\uparrow$ iff $t = S(0, y \in y, \emptyset)$; if t is a location term we have $t\!\uparrow$ iff $t = (0, \varphi_0, 1)$.

If $s \mathrel{\tilde{<}} s'$ with $\alpha(s) = \alpha(s')$ and φ is quantifier-free and determined by s, then φ is determined by s' and $L_s \models \varphi$ iff $L_{s'} \models \varphi$.

The concept of "determined" is needed so that a term which is undefined cannot become defined for a bigger location on the same level, thereby changing truth values of formulas. For level changes we also get persistence provided terms are translated (as indicated above). From now on those translations won't be mentioned.

If $s \mathrel{\tilde{<}} s'$ and φ is a Σ_1-formula with $L_s \models \varphi$, then $L_{s'} \models \varphi$.

If φ is a Σ_1-formula with $L_s \models \varphi$ and s is a limit location, then there is an $s' \mathrel{\tilde{<}} s$ s. t. $L_{s'} \models \varphi$.

Let $\pi\colon L_s \to L_t$ be a structure-preserving map with s, t limit locations. π is Σ_1-preserving iff $\operatorname{range}\pi$ is Σ_1-closed (i. e., $\operatorname{range}\pi = L_t^*\{\operatorname{range}\pi\}$): Clearly if $\operatorname{range}\pi$ is Σ_1-closed then π is Σ_1-preserving; for the other direction just note that if you have a witness for a Σ_1-formula then it is Σ_1 to say there is a smaller one.

Lemma 4 *Let s be a location and $s_0 = (\alpha_0, \varphi_{n_0}, p_0) \mathrel{\tilde{<}} s$. For every term in the language for L_{s_0} we have a QFF in the language for L_s (uniformly definable using α_0 and p_0 as parameters and the free variables of the term) which is true in L_s for an L_{s_0}-assignment of the variables iff the term is defined in L_{s_0} with the same assignment.*

Proof This is done by induction on the complexity of a term (everything is evaluated according to the assignment). We write $\mathrm{def}_{s_0}(t)$ for "t is a defined term in L_{s_0}". For a variable v_i, set terms x, y and a location term t we have:

— $L_s \models \mathrm{def}_{s_0}(v_i)$.

— $L_s \models \mathrm{def}_{s_0}(\alpha(t))$ iff $L_s \models \mathrm{def}_{s_0}(t) \wedge \alpha(t) < \alpha_0$.

— $L_s \models \mathrm{def}_{s_0}(\vec{x}(t))$ iff $L_s \models \mathrm{def}_{s_0}(t)$.

— $L_s \models \mathrm{def}_{s_0}((x, \varphi_n, y))$ iff $L_s \models \mathrm{def}_{s_0}(x) \wedge \mathrm{def}_{s_0}(y) \wedge (x, \varphi_n, y)\downarrow$.

— $L_s \models \mathrm{def}_{s_0}((\alpha_0, \varphi_n, y))$ iff
$L_s \models \mathrm{def}_{s_0}(y) \wedge (\alpha_0, \varphi_n, y)\downarrow \wedge (\alpha_0, \varphi_n, y) \mathrel{\widetilde{<}} s_0$.

— $L_s \models \mathrm{def}_{s_0}(N(x))$ iff $L_s \models \mathrm{def}_{s_0}(x)$.

— $L_s \models \mathrm{def}_{s_0}(I(t))$ iff $L_s \models \mathrm{def}_{s_0}(t) \wedge \alpha(t) < \alpha_0$.

— $L_s \models \mathrm{def}_{s_0}(S(t))$ iff $L_s \models \mathrm{def}_{s_0}(t) \wedge S(t)\downarrow$. $\dashv$

Lemma 5 *Let s be a location and $s_0 = (\alpha_0, \varphi_{n_0}, p_0) \mathrel{\widetilde{<}} s$. For every term in the language for L_{s_0} we have a QFF in the language for L_s (uniformly definable using α_0 and p_0 as parameters and the free variables of the term) which is true in L_s for an L_{s_0}-assignment of the variables iff the term is determined by s_0 with the same assignment.*

Proof As in the previous lemma, this is done by induction on the complexity of a term where everything is evaluated according to the assignment. We write $\mathrm{det}_{s_0}(t)$ for "t is determined by s_0". For a variable v_i, set terms x, y and a location term t we have:

— $L_s \models \mathrm{det}_{s_0}(v_i)$.

— $L_s \models \mathrm{det}_{s_0}(\alpha(t))$ iff $L_s \models \mathrm{det}_{s_0}(t)$.

— $L_s \models \mathrm{det}_{s_0}(\vec{x}(t))$ iff $L_s \models \mathrm{det}_{s_0}(t)$.

— $L_s \models \mathrm{det}_{s_0}((x, \varphi_n, y))$ iff $L_s \models \mathrm{det}_{s_0}(x) \wedge \mathrm{det}_{s_0}(y)$.

— $L_s \models \mathrm{det}_{s_0}((\alpha_0, \varphi_n, y))$ iff
$L_s \models \mathrm{det}_{s_0}(y) \wedge \big((\alpha_0, \varphi_n, y)\downarrow \to (\alpha_0, \varphi_n, y) \mathrel{\widetilde{<}} s_0\big)$.

— $L_s \models \mathrm{det}_{s_0}(N(x))$ iff $L_s \models \mathrm{det}_{s_0}(x)$.

— $L_s \models \mathrm{det}_{s_0}(I(t))$ iff $L_s \models \mathrm{det}_{s_0}(t)$.

— $L_s \models \det_{s_0}(S(t))$ iff $L_s \models \det_{s_0}(t)$. $\dashv$

Corollary 6 *Let s be a location and $s_0 = (\alpha_0, \varphi_{n_0}, p_0) \mathrel{\widetilde{<}} s$. For every QFF φ in the language for L_{s_0} there is a QFF φ' in the language for L_s (uniformly definable) which is true in L_s for an L_{s_0}-assignment of the variables iff φ is true in L_{s_0} with the same assignment.*

Proof Using $\det_{s_0}$ for every term in φ we can check that φ is determined by s_0. Then by induction on the complexity of the formula using def_{s_0} and $\det_{s_0}$ we express the truth of φ. $\dashv$

Definition 7 (Type) Let s be a location and $\vec{x}, \vec{p} \in L_{\alpha(s)}$. Define:

$$\begin{aligned}\mathrm{Type}(s, \vec{x}, \vec{p}) = \{(0, \tau_0, \tau_1) \mid \tau_0,\ \tau_1 \text{ terms},\ L_s \models \tau_0(\vec{x}, \vec{p}) = \tau_1(\vec{x}, \vec{p})\} \cup \\ \cup \{(1, \tau_0, \tau_1) \mid \tau_0,\ \tau_1 \text{ terms},\ L_s \models \tau_0(\vec{x}, \vec{p}) \in \tau_1(\vec{x}, \vec{p})\}\end{aligned}$$

Lemma 8 (Type Preservation) *Let $\pi\colon L_s \to L_t$ be a Σ_1-preserving map, $s_0 \mathrel{\widetilde{\leq}} s_1 \mathrel{\widetilde{\leq}} s$, $\vec{p_0} \in L_{s_0}$, $\vec{p_1} \in L_{s_1}$, s_0, s_1 limit locations, and $\alpha \leq \alpha(s_0)$. Then:*

$$\begin{aligned}&\forall \vec{x} \in \alpha\ \mathrm{Type}\,(s_0, \vec{x}, \vec{p_0}) = \mathrm{Type}\,(s_1, \vec{x}, \vec{p_1}) \ \textit{ iff} \\ &\forall \vec{x} \in \pi(\alpha)\ \mathrm{Type}\,(\pi(s_0), \vec{x}, \pi(\vec{p_0})) = \mathrm{Type}\,(\pi(s_1), \vec{x}, \pi(\vec{p_1}))\end{aligned}$$

Proof $\forall$ is preserved downwards (note the implicit $\forall$ quantification over terms). So it remains to show, that the upward direction is preserved. Let

$$\neg\, \forall \vec{x} \in \pi(\alpha)\ \mathrm{Type}\,(\pi(s_0), \vec{x}, \pi(\vec{p_0})) = \mathrm{Type}\,(\pi(s_1), \vec{x}, \pi(\vec{p_1}))$$

Equivalently:

$$\exists \vec{x} \in \pi(\alpha)\ \mathrm{Type}\,(\pi(s_0), \vec{x}, \pi(\vec{p_0})) \neq \mathrm{Type}\,(\pi(s_1), \vec{x}, \pi(\vec{p_1}))$$

Hence there are terms τ_0, τ_1 which witness this inequality, e.g., $(0, \tau_0, \tau_1)$ is in the right Type but not in the left one. So using corollary 6 we can write:

$$\begin{aligned}L_t \models \exists \vec{x} \in \pi(\alpha)\ \big(L_{\pi(s_0)} \models (\tau_0(\vec{x}, \pi(\vec{p_0})) \neq \tau_1(\vec{x}, \pi(\vec{p_0}))) \wedge \\ \wedge\ L_{\pi(s_1)} \models (\tau_0(\vec{x}, \pi(\vec{p_1})) = \tau_1(\vec{x}, \pi(\vec{p_1})))\big)\end{aligned}$$

This is a Σ_1-statement and therefore preserved. $\dashv$

Corollary 9 *With the hypotheses of the lemma we get:*

$$\begin{aligned}&L_{s_0}\{\alpha \cup \vec{p_0}\} \cong L_{s_1}\{\alpha \cup \vec{p_1}\} \ \textit{ iff} \\ &L_{\pi(s_0)}\{\pi(\alpha) \cup \pi(\vec{p_0})\} \cong L_{\pi(s_1)}\{\pi(\alpha) \cup \pi(\vec{p_1})\}\end{aligned}$$

Proof First assume $\pi_1 \colon L_{s_0}\{\alpha \cup \vec{p_0}\} \cong L_{s_1}\{\alpha \cup \vec{p_1}\}$. π_1 is structure preserving and hence preserves determinedness of terms. Therefore, we have $\mathrm{Type}\,(s_0, \vec{x}, \vec{p_0}) = \mathrm{Type}\,(s_1, \vec{x}, \vec{p_1})$ for all $\vec{x} \in \alpha$. Now apply type preservation along π to get $\mathrm{Type}\,(\pi(s_0), \vec{x}, \pi(\vec{p_0})) = \mathrm{Type}\,(\pi(s_1), \vec{x}, \pi(\vec{p_1}))$ for all $\vec{x} \in \pi(\alpha)$. This shows we have an isomorphism as required: $L_{\pi(s_0)}\{\pi(\alpha) \cup \pi(\vec{p_0})\} \cong L_{\pi(s_1)}\{\pi(\alpha) \cup \pi(\vec{p_1})\}$. The same argument works for the other direction. $\dashv$

1 Morasses in L

Gap-1

Definition 10 (Gap-1-Morass) An $(\omega_1, 1)$-*morass* (morass, from now on) is a structure of the form $\left\langle S^1, \mathrel{-\!\!\!3}, (\pi_{\sigma\tau})_{\sigma \mathrel{-\!\!\!3} \tau} \right\rangle$ with

$$\begin{array}{lcl} S^0,\ S^1 & \subset & \omega_2, \\ \gamma_\sigma & \in & S^0 \text{ for } \sigma \in S^1, \\ S_\gamma & := & \left\{ \sigma \in S^1 \mid \gamma_\sigma = \gamma \right\} \text{ for } \gamma \in S^0, \\ S^0 & = & \left\{ \gamma_\sigma \mid \sigma \in S^1 \right\}, \text{ and} \\ \prec,\ \mathrel{-\!\!\!3} & \subset & S^1 \times S^1. \end{array}$$

Let $\mathrel{-\!\!\!3}$ be a partial ordering on S^1.

(M0) i) $\forall \gamma \in S^0 \cap \omega_1\ S_\gamma$ closed

ii) S_{ω_1} club in ω_2

iii) $\omega_1 = \sup(S^0 \cap \omega_1) \in S^0$

iv) $\mathrel{-\!\!\!3}$ is a tree-ordering on S^1

(M1) If $\sigma \mathrel{-\!\!\!3} \tau$ then

i) $\pi_{\sigma\tau} \colon \sigma + 1 \to \tau + 1$, $\pi_{\sigma\tau} \restriction \gamma_\sigma = \mathrm{id} \restriction \gamma_\sigma$, $\gamma_\sigma < \pi_{\sigma\tau}(\gamma_\sigma) = \gamma_\tau$, $\pi_{\sigma\tau}(\sigma) = \tau$

ii) $\pi_{\sigma\tau}$ is order-preserving with $\pi_{\sigma\tau}^{-1}\left[S_{\gamma_\tau} \cap (\tau + 1)\right] = S_{\gamma_\sigma} \cap (\sigma + 1)$.

iii) For all $\nu \preceq \sigma$, ν is $\prec$-minimal, successor, limit iff $\pi_{\sigma\tau}(\nu)$ is $\prec$-minimal, successor, limit, respectively. In the successor case also the immediate predecessor is preserved.

(M2) Let $\sigma \mathrel{-\!\!\!3} \tau$, $\bar\sigma \prec \sigma$, and $\bar\tau := \pi_{\sigma\tau}(\bar\sigma)$, then $\bar\sigma \mathrel{-\!\!\!3} \bar\tau$ via $\pi_{\bar\sigma\bar\tau} = \pi_{\sigma\tau} \restriction (\bar\sigma + 1)$.

(M3) For $\tau \in S^1$ $\{\gamma_\sigma \mid \sigma \mathrel{-\!\!\!3} \tau\}$ closed in γ_τ.

(M4) If τ is not $\prec$-maximal then $\{\gamma_\sigma \mid \sigma \mathrel{-\!\!\!3} \tau\}$ cofinal in γ_τ.

(M5) If $\{\gamma_\sigma \mid \sigma \mathrel{-\!\!\!3} \tau\}$ is unbounded in γ_τ, then $\tau = \bigcup\limits_{\sigma \mathrel{-\!\!\!3} \tau} \pi_{\sigma\tau}[\sigma]$.

(M6) If $\sigma \mathrel{-\!\!\!3} \tau$, σ a $\prec$-limit, and $\lambda := \sup \operatorname{range} \pi_{\sigma\tau} \restriction \sigma < \tau$, then $\sigma \mathrel{-\!\!\!3} \lambda$ with $\pi_{\sigma\lambda} \restriction \sigma = \pi_{\sigma\tau} \restriction \sigma$.

(M7) If $\sigma \mathrel{-\!\!\!3} \tau$, σ a $\prec$-limit, and $\tau = \sup \operatorname{range} \pi_{\sigma\tau} \restriction \sigma$, then for $\alpha \in S^0$, if $\forall \bar\sigma \prec \sigma\ \exists \bar\upsilon \in S_\alpha\ \bar\sigma \mathrel{-\!\!\!3} \bar\upsilon \mathrel{-\!\!\!3} \pi_{\sigma\tau}(\bar\sigma)$ then $\exists \upsilon \in S_\alpha\ \sigma \mathrel{-\!\!\!3} \upsilon \mathrel{-\!\!\!3} \tau$.

Definition 11 $\sigma < \omega_2$ is called $(\omega_1, 1)$-*morass point* (morass point, from now on) iff $\sigma = \bigcup \{\mu < \sigma \mid L_\mu \models ZF^-\}$, and $L_\sigma \models \exists! \gamma \in \text{Card } \gamma > \aleph_0$. In this case, let γ_σ be this unique ordinal. Let $S^1 = \{\sigma < \omega_2 \mid \sigma \text{ morass point}\}$ and $S^0 := \left\{\gamma_\sigma \mid \sigma \in S^1\right\}$. For $\sigma, \tau \in S^1$ define $\sigma \prec \tau$ iff $\sigma < \tau \wedge \gamma_\sigma = \gamma_\tau$.

For $\sigma \in S^1$ let $s(\sigma)$ be the $\widetilde{\leq}$-least location s s. t. there is a $p \in {}^{<\omega}L_{\alpha(s)}$ with $L_s \{\gamma_\sigma \cup p\} \cap \sigma$ cofinal in σ (we say: $L_s \{\gamma_\sigma \cup p\}$ collapses σ); in this case let p_σ be the $<^*$-least such.

Define the partial ordering $\multimap$ on S^1 by letting $\sigma \multimap \tau$ iff there exists $\pi \colon L_{s(\sigma)} \to L_{s(\tau)}$ with:

i) π is Σ_1-preserving.

ii) $\pi \restriction \gamma_\sigma = \text{id} \restriction \gamma_\sigma$, $\gamma_\sigma < \pi(\gamma_\sigma) = \gamma_\tau$, $\tau = \pi(\sigma)$, $p_\tau \in \text{range}\, \pi$
(define $\pi(\sigma) = \tau$ if $\sigma \notin \text{dom}\, \pi$)

iii) If τ is a $\prec$-successor with immediate predecessor τ', then $\tau' \in \text{range}\, \pi$.

Lemma 12

i) $\sigma \subset L_{s(\sigma)} \{\gamma_\sigma \cup p_\sigma\}$

ii) $L_{s(\sigma)} \{\sigma \cup p_\sigma\} = L_{s(\sigma)}$

iii) $L_{s(\sigma)} \{\gamma_\sigma \cup p_\sigma\} = L_{s(\sigma)}$

iv) The map $\pi \colon L_{s(\sigma)} \to L_{s(\tau)}$, *if exists, is uniquely determined.*

v) $\pi(p_\sigma) = p_\tau$

vi) If τ' *is the immediate* $\prec$*-predecessor of* τ, *then* $\pi^{-1}(\tau')$ *is the immediate* $\prec$*-predecessor of* σ.

Proof For i) assume $\xi \in L_{s(\sigma)} \{\gamma_\sigma \cup p_\sigma\} \cap \sigma$. Let $\eta \in L_{s(\sigma)} \{\gamma_\sigma \cup p_\sigma\} \cap \sigma$ s. t. $\exists f \in L_\eta\, f \colon \gamma_\sigma \leftrightarrow \xi$. In particular, $S\left(\eta, v_0 \colon \aleph_1 \leftrightarrow v_1, \langle \xi \rangle\right)$ is such a map. Therefore, $\xi = \text{range}\, f \subset L_{s(\sigma)} \{\gamma_\sigma \cup p_\sigma\}$. Using that the hull is cofinal in σ we have that σ actually is a subset.

For ii) consider $L_{s(\sigma)} \{\sigma \cup p_\sigma\} \cong L_{\bar{s}} \{\sigma \cup \bar{p}\} = L_{\bar{s}}$. Then $\bar{s}, \bar{p}$ satisfy the definition of $s(\sigma), p_\sigma$; by minimality we have $s(\sigma) = \bar{s}$ and $p_\sigma = \bar{p}$.

iii) follows from i) and ii). Now iv) is clear.

For v) note, that $p_\tau \in \text{range}\, \pi$. By definition $\pi(p_\sigma) \in L_{s(\tau)} \{\gamma_\tau \cup p_\tau\}$. Using Σ_1-preservation, we get $p_\sigma \in L_{s(\sigma)} \left\{\gamma_\sigma \cup \pi^{-1}(p_\tau)\right\}$ and hence $L_{s(\sigma)} = L_{s(\sigma)} \left\{\gamma_\sigma \cup \pi^{-1}(p_\tau)\right\}$. Therefore, $p_\sigma \leq^* \pi^{-1}(p_\tau)$. Assume for contradiction

that this is strict. Then we get $\pi(p_\sigma) <^* p_\tau$. But $p_\tau \in L_{s(\tau)}\{\gamma_\sigma \cup \pi(p_\sigma)\} \subset L_{s(\tau)}\{\gamma_\tau \cup \pi(p_\sigma)\} = L_{s(\tau)}$ contradicting the minimality of p_τ.

For vi) assume for contradiction, that there is $\sigma' \prec \sigma$ with $\sigma' > \pi^{-1}(\tau')$. Let $\eta < \sigma$ be large enough s. t. $L_\eta \models ZF^- \wedge \sigma'$ morass point. Now $\pi \restriction L_\eta$ is elementary and, therefore, $L_{\pi(\eta)} \models \pi_{\sigma\tau}(\sigma')$ morass point. But $\pi(\sigma') > \tau'$, contradicting that τ' is the immediate predecessor of τ. $\dashv$

Definition 13 (morass map) For $\sigma \multimap\!\!\!3\; \tau$, let $\pi_{\sigma\tau}$ be the unique map from the previous lemma. The actual morass map to satisfy the morass axioms will be $(\pi_{\sigma\tau}\restriction\sigma) \cup \{(\sigma,\tau)\}$ (note that $\pi_{\sigma\tau}(\sigma) = \tau$ if $\sigma \in \operatorname{dom}\pi_{\sigma\tau}$), but we will write $\pi_{\sigma\tau}$ for both maps and work with the underlying map only.

Theorem 14 $\langle S^1, \multimap\!\!\!3, (\pi_{\sigma\tau})_{\sigma \multimap\!\!\!3 \tau}\rangle$ *as defined above is an* $(\omega, 1)$*-morass.*

Proof For **(M0)** the first three assertions are clear. To see that $\multimap\!\!\!3$ forms a tree-ordering, presume $\pi_{\sigma\tau}$ and $\pi_{\upsilon\tau}$ are morass maps with $\sigma < \upsilon$; then $\pi_{\sigma\upsilon} = \pi_{\upsilon\tau}^{-1} \circ \pi_{\sigma\tau}$ is as required.

For **(M1)** the first assertion is as defined. For ii) note first that morass points $\bar\nu \leq \sigma$ are mapped to morass points $\leq \tau$: This is clear for $\bar\nu = \sigma$; to see this for $\bar\nu < \sigma$ choose a suitable restriction of $\pi_{\sigma\tau}$ to a ZF^--model. Clearly, the map is order-preserving. The next properties for morass points below the top are immediate from the preceding, again by elementarity. For morass point at the top we use lemma 12 vi).

To see **(M2)** first note that by (M1) $\bar\tau$ is a morass point. Using that L_σ is a limit of ZF^--models find $\eta < \sigma$ s. t. $L_{s(\bar\sigma)}$ and $p_{\bar\sigma}$ are definable in L_η from the parameter $\bar\sigma$. Hence $\pi_{\sigma\tau} \restriction L_\eta$ is elementary and, therefore, maps $L_{s(\bar\sigma)}$ into $L_{s(\bar\tau)}$ and $p_{\bar\sigma}$ onto $p_{\bar\tau}$. Then $\pi_{\bar\sigma\bar\tau}$ is as required.

For **(M3)** let $\tau \in S^1$ and $\bar\alpha < \gamma_\tau$ a limit point of $\{\gamma_\sigma \mid \sigma \multimap\!\!\!3\; \tau\}$. By condensation let $\pi\colon L_{s(\tau)}\{\bar\alpha \cup p_\tau\} \cong L_{\bar s}$ and $\bar\tau = \pi(\tau)$, $\bar p = \pi(p_\tau)$. Note that $L_{s(\tau)}\{\bar\alpha \cup p_\tau\} \cap \gamma_\tau = \bar\alpha$, since $\bar\alpha$ is the limit of $L_{s(\tau)}\{\gamma_\sigma \cup p_\tau\} \cap \gamma_\tau = \gamma_\sigma < \bar\alpha$.

We show $\bar s = s(\bar\tau)$: Clearly $s(\bar\tau) \mathrel{\tilde{\leq}} \bar s$, since $L_{\bar s} = L_{\bar s}\{\bar\alpha \cup \bar p\}$ cofinal in $\bar\tau$. Now assume for contradiction that $s(\bar\tau) \mathrel{\tilde{<}} \bar s$. Let $\pi_\sigma = \pi \circ \pi_{\sigma\tau}$ for $\sigma \in \{\sigma \multimap\!\!\!3\; \tau \mid \gamma_\sigma < \bar\alpha\}$. Choose σ large enough s. t. exist $\tilde s, \tilde p \in L_{s(\sigma)}$ with $s(\bar\tau) = \pi_\sigma(\tilde s)$ and $p_{\bar\tau} = \pi_\sigma(\tilde p)$. By $s(\bar\tau) \mathrel{\tilde{<}} \bar s$ we have $\tilde s \mathrel{\tilde{<}} s(\sigma)$ and hence $L_{\tilde s}\{\gamma_\sigma \cup \tilde p\}$ bounded in σ, say by β. But this bound is preserved by $\pi_{\sigma\tau}$ and by π (hence by π_σ);

therefore, we get that $L_{s(\bar\tau)}\,\{\bar\alpha \cup p_{\bar\tau}\} \cap \bar\tau$ is bounded by $\pi_\sigma(\beta) < \bar\tau$ which contradicts the definition of $s(\bar\tau)$ and $p_{\bar\tau}$.

To see that $\pi^{-1}\colon L_{s(\bar\tau)} \to L_{s(\tau)}$ is a morass map and hence $\bar\tau \mathrel{-\!\!\!\!3} \tau$ with $\gamma_{\bar\tau} = \bar\alpha$, we need to show, that π^{-1} preserves Σ_1; the other properties follow by definition, for p_τ and the predecessor of τ (if any) note that $\operatorname{dom}\pi$ contains the ranges of morass maps as subsets.

As a collapsing map, π^{-1} is structure-preserving. Σ_1 is preserved upwards. Now assume, we have a Σ_1-formula in $L_{s(\tau)}$. It is preserved downwards by morass maps $\pi_{\sigma\tau}$ for $\sigma \in \{\sigma \mathrel{-\!\!\!\!3} \tau \mid \gamma_\sigma < \bar\alpha\}$ and hence has a witness in $\operatorname{range}\pi_{\sigma\tau} \subset \operatorname{dom}\pi$.

For the proof of **(M4)** let $\upsilon \in S_{\gamma_\tau}$ with $\tau < \upsilon$. Let $\alpha < \gamma_\tau$ be arbitrary and η between τ and υ s.t. $L_{s(\tau)} \in L_\eta$ and $L_\eta \models ZF^-$. Let $X \prec L_\eta$ s.t. $L_{s(\tau)}\,\{\alpha \cup p_\tau\} \cup \{\tau\} \subset X$ and $\bar\alpha := X \cap \gamma_\tau \in \gamma_\tau$. Let $\pi\colon X \cong L_{\bar\eta}$, $\sigma = \pi(\tau)$, and $\bar p = \pi(p_\tau)$. So σ is a morass point and $\pi^{-1} \restriction L_{s(\sigma)}\colon L_{s(\sigma)} \to L_{s(\tau)}$ is elementary and, therefore, a morass map. Hence $\sigma \mathrel{-\!\!\!\!3} \tau$ and $\alpha \leq \gamma_\sigma = \bar\alpha$.

For **(M5)** consider $\xi \in \tau \in S^1$ and $L_{s(\tau)} = L_{s(\tau)}\,\{\gamma_\tau \cup p_\tau\}$. By cofinality there exists a $\sigma \mathrel{-\!\!\!\!3} \tau$ with $\xi \in L_{s(\tau)}\,\{\gamma_\sigma \cup p_\tau\} = \operatorname{range}\pi_{\sigma\tau}$.

For **(M6)** let $\tilde s = \tilde\leqq$-lub $\{\pi_{\sigma\tau}(t) \mid t \mathrel{\tilde<} s(\sigma)\}$. We show $L_{\tilde s}\,\{\gamma_\tau \cup p_\tau\} \cap \tau = \lambda$: First assume $\lambda_0 \in \lambda$, then there is λ_1 with $\lambda_0 < \lambda_1 < \lambda$ and $\lambda_1 = \pi_{\sigma\tau}(\bar\lambda_1)$. Then $L_\sigma \models \bar\lambda_1 \leq \gamma_\sigma$, hence exists $\bar f \in L_\sigma$ s.t. $\bar f\colon \gamma_\sigma \twoheadrightarrow \bar\lambda_1$, in particular $\bar f \in L_{s(\sigma)}\,\{\gamma_\sigma \cup p_\sigma\}$. As in lemma 12, also $\bar f \in L_t\,\{\gamma_\sigma \cup p_\sigma\}$ for some $t \mathrel{\tilde<} s(\sigma)$ with $\alpha(t) < \alpha(s(\sigma))$. Let $f = \pi_{\sigma\tau}(\bar f) \in L_{\pi_{\sigma\tau}(t)}\,\{\gamma_\tau \cup p_\tau\}$, then $f\colon \gamma_\tau \twoheadrightarrow \lambda_1$, so $\lambda_0 \in \operatorname{range} f$, hence $\lambda_0 \in L_{\tilde s}\,\{\gamma_\tau \cup p_\tau\}$. On the other hand assume $\lambda_0 \in L_{\tilde s}\,\{\gamma_\tau \cup p_\tau\} \cap \tau$, then there is a $t \mathrel{\tilde<} s(\sigma)$ s.t. $\lambda_0 \in L_{\pi_{\sigma\tau}(t)}\,\{\gamma_\tau \cup p_\tau\}$. But $L_t\,\{\gamma_\sigma \cup p_\sigma\} \cap \sigma$ is bounded below σ (by β say), since $t \mathrel{\tilde<} s(\sigma)$, hence also $L_{\pi_{\sigma\tau}(t)}\,\{\gamma_\tau \cup p_\tau\} \cap \tau$ is bounded below τ, namely by $\pi_{\sigma\tau}(\beta) < \lambda$. So $\lambda_0 \in \lambda$ as required.

Let $\pi\colon L_{\tilde s}\,\{\gamma_\tau \cup p_\tau\} \cong L_{s_0}$ and $p_0 = \pi(p_\tau)$ (then $\lambda = \pi(\tau)$). Note that $\lambda \in S_{\gamma_\tau}$. We show $L_{s_0}\,\{\gamma_\tau \cup p_0\} = L_{s(\lambda)}\,\{\gamma_\tau \cup p_\lambda\}$:

$s_0 = s(\lambda)$: First note that s_0 singularizes λ, so $s(\lambda) \mathrel{\tilde\leq} s_0$. Assume for contradiction that s_0 is strictly greater. Then $p_\lambda \in L_{s_0}\,\{\gamma_\tau \cup p_0\}$, hence $p_\lambda \in L_{s_1}\,\{\gamma_\tau \cup p_0\}$ where $s(\lambda) \mathrel{\tilde<} s_1 \mathrel{\tilde<} s_0$ (using that s_0 is a limit location). Since $L_{s(\lambda)}\,\{\gamma_\tau \cup p_\lambda\} \subset L_{s_1}\,\{\gamma_\tau \cup p_0\}$, s_1 singularizes λ. By definition of s_0, $\pi^{-1}(s_1) \mathrel{\tilde<} \tilde s$. Further, by definition of $\tilde s$, there is a $t \mathrel{\tilde<} s(\sigma)$ s.t. $\pi^{-1}(s_1) \mathrel{\tilde\leq}$

$\pi_{\sigma\tau}(t)$. By minimality of $s(\sigma)$, $L_t\{\gamma_\sigma \cup p_\sigma\} \cap \sigma$ is bounded below σ (by β say). Hence $L_{\pi_{\sigma\tau}(t)}\{\gamma_\tau \cup p_\tau\} \cap \tau$ is bounded below τ (by $\pi_{\sigma\tau}(\beta)$). Since $\pi^{-1}(s_1) \mathrel{\tilde{\leq}} \pi_{\sigma\tau}(t)$, $L_{\pi^{-1}(s_1)}\{\gamma_\tau \cup p_\tau\} \cap \tau$ is bounded below τ (still by $\pi_{\sigma\tau}(\beta)$). Apply π: $L_{s_1}\{\gamma_\tau \cup p_0\} \cap \lambda$ is bounded below λ (by $\pi \circ \pi_{\sigma\tau}(\beta)$), contradiction.

$p_0 = p_\lambda$: $L_{s(\lambda)} = L_{s(\lambda)}\{\gamma_\tau \cup p_0\}$ is cofinal in λ (as above using $s_0 = s(\lambda)$). Therefore, $p_\lambda \leq^* p_0$. Assume for contradiction that p_0 is strictly greater, then using $p_0 \in L_{s(\lambda)} = L_{s(\lambda)}\{\gamma_\tau \cup p_\lambda\}$ and applying π^{-1} we get $\pi^{-1}(p_\lambda) <^* p_\tau \in L_{\tilde{s}}\left\{\gamma_\tau \cup \pi^{-1}(p_\lambda)\right\} \subset L_{s(\tau)}\left\{\gamma_\tau \cup \pi^{-1}(p_\lambda)\right\}$. Therefore, $L_{s(\tau)} = L_{s(\tau)}\{\gamma_\tau \cup p_\tau\} = L_{s(\tau)}\left\{\gamma_\tau \cup \pi^{-1}(p_\lambda)\right\}$ contradicting the minimality of p_τ.

Let $\pi_0 = \pi \circ \pi_{\sigma\tau} \colon L_{s(\sigma)} \to L_{s(\lambda)}$. π_0 is well-defined as $\operatorname{range} \pi_{\sigma\tau} = L_{\tilde{s}}\{\gamma_\sigma \cup p_\tau\} \subset \operatorname{dom} \pi$. Further, $\pi_0(\sigma) = \lambda$ and $\pi_0(p_\sigma) = p_\lambda$. Since λ is a $\prec$-limit, property iii) of the morass map definition is vacuous. Finally, π_0 is Σ_1-preserving: First note that π_0 is structure-preserving. Σ_1 formulas are preserved by π_0 upwards, by π upwards (from $L_{s(\lambda)}$ to $L_{\tilde{s}}\{\gamma_\tau \cup p_\tau\}$), and by $\pi_{\sigma\tau}$ downwards, hence by π_0 both ways. Now $\pi_0 = \pi_{\sigma\lambda}$ is a morass map, hence $\sigma \multimap \lambda$ as required.

For **(M7)** we first show that $L_{s(\tau)}\{\alpha \cup p_\tau\} \cap \gamma_\tau = \alpha$, clearly α is a subset of the left side. For the other direction note that as in (M6) we have $s(\tau) = \tilde{\leq}\text{-lub}\left\{\pi_{\sigma\tau}(t) \mid t \mathrel{\tilde{<}} s(\sigma)\right\}$ (define $\tilde{s}$ and note that $\tilde{s} = s(\tau)$). Let $\xi \in L_{s(\tau)}\{\alpha \cup p_\tau\} \cap \gamma_\tau$, then there is $s_0 \mathrel{\tilde{<}} s(\sigma)$ s. t. $\xi \in L_{\pi_{\sigma\tau}(s_0)}\{\alpha \cup p_\tau\} \cap \gamma_\tau$. Working downstairs we have that $L_{s_0}\{\gamma_\sigma \cup p_\sigma\}$ does not collapse σ (by minimality of $s(\sigma) \mathrel{\tilde{>}} s_0$). Let $\pi_0 \colon L_{\bar{s}} = L_{\bar{s}}\{\gamma_\sigma \cup \bar{p}\} \cong L_{s_0}\{\gamma_\sigma \cup p_\sigma\}$ where $\bar{p} = \pi_0^{-1}(p_\sigma)$. Then $\sigma' := \pi_0^{-1}(\sigma) < \sigma$. $L_{\bar{s}}$ cannot collapse σ', else there would be a map from γ_σ onto σ' and hence a map from γ_σ onto σ in $L_{s_0}\{\gamma_\sigma \cup p_\sigma\}$. Therefore, $L_{\bar{s}} \models \operatorname{Card} \sigma'$ and $L_\sigma \models \neg \operatorname{Card} \sigma'$, hence $L_{\bar{s}} \in L_\sigma$. Now, σ is a $\prec$-limit, so there is $\bar{\sigma} \prec \sigma$ s. t. $L_{\bar{s}}, \bar{p} \in L_{s(\bar{\sigma})} = L_{s(\bar{\sigma})}\{\gamma_\sigma \cup p_{\bar{\sigma}}\}$.

Using lemma 8 (type preservation) we shift the isomorphism π_0 to $L_{s(\tau)}$:

$$\text{“}\pi_{\sigma\tau}(\pi_0)\text{”} \colon L_{\pi_{\sigma\tau}(\bar{s})}\{\gamma_\tau \cup \pi_{\sigma\tau}(\bar{p})\} \cong L_{\pi_{\sigma\tau}(s_0)}\{\gamma_\tau \cup p_\tau\}$$

We started with $\xi \in L_{\pi_{\sigma\tau}(s_0)}\{\alpha \cup p_\tau\} \cap \gamma_\tau$. Now we apply the isomorphism and infer $\xi \in L_{\pi_{\sigma\tau}(\bar{s})}\{\alpha \cup \pi_{\sigma\tau}(\bar{p})\} \cap \gamma_\tau$ (since $\xi < \gamma_\tau$ it is not moved). Further, $L_{\pi_{\sigma\tau}(\bar{s})}\{\alpha \cup \pi_{\sigma\tau}(\bar{p})\} \cap \gamma_\tau \subset L_{s(\pi_{\sigma\tau}(\bar{\sigma}))}\left\{\alpha \cup p_{\pi_{\sigma\tau}(\bar{\sigma})}\right\} \cap \gamma_\tau = \alpha$, where the former holds since $\pi_{\sigma\tau}(\bar{p}) \in L_{\pi_{\sigma\tau}(\bar{\sigma})}\left\{\gamma_\sigma \cup p_{\pi_{\sigma\tau}(\bar{\sigma})}\right\}$ and $\pi_{\sigma\tau}(\bar{s}) \mathrel{\tilde{<}} s(\pi_{\sigma\tau}(\bar{\sigma}))$ and the latter holds by $\bar{\sigma} \multimap \bar{\upsilon} \multimap \pi_{\sigma\tau}(\bar{\sigma})$ for some $\bar{\upsilon} \in S_\alpha$. Hence $\xi \in \alpha$ as desired.

Now we define $\pi\colon L_{s(\tau)}\{\alpha \cup p_\tau\} \cong L_{s'}\{\alpha \cup p'\} = L_{s'}$ where $p' := \pi(p_\tau)$, $\upsilon := \pi(\tau)$. By the previous argument we have $\pi^{-1}(\alpha) = \gamma_\tau$. Using the system of morass maps we have $\upsilon \in S_\alpha$.

We have to show $s' = s(\upsilon)$: $L_{s'} = L_{s'}\{\alpha \cup p'\}$ collapses υ, hence $s(\upsilon) \mathrel{\widetilde{\leq}} s'$, assume $s(\upsilon) \mathrel{\widetilde{<}} s'$. Since $p_\upsilon \in L_{s'}$ we have that there is an s_0 s. t. $s(\upsilon) \mathrel{\widetilde{\leq}} s_0 \mathrel{\widetilde{<}} s'$ and $p_\upsilon \in L_{s_0}\{\alpha \cup p'\}$. Since $\pi_{\sigma\tau}$ and π map locations cofinally this is also true for $\pi_0 := \pi \circ \pi_{\sigma\tau}$ (locations $\widetilde{<}\, s(\sigma)$ are mapped to locations $\widetilde{<}\, s'$). Hence wlog $s_0 = \pi_0(\bar{s}_0)$ where $\bar{s}_0 \mathrel{\widetilde{<}} s(\sigma)$. Therefore, $L_{s(\sigma)} \models$ "$L_{\bar{s}_0}\{\gamma_\sigma \cup p_\sigma\}$ is bounded below σ". This is preserved by $\pi_{\sigma\tau}$: $L_{s(\tau)} \models$ "$L_{\pi_{\sigma\tau}(\bar{s}_0)}\{\gamma_\tau \cup p_\tau\}$ is bounded below τ". Finally, this is preserved by π downwards: $L_{s'} \models$ "$L_{s_0}\{\alpha \cup p'\}$ is bounded below υ", contradicting the definition of $s(\upsilon) \mathrel{\widetilde{\leq}} s_0$.

Finally, we have to show that π^{-1} is Σ_1-preserving, then $\pi^{-1} = \pi_{\upsilon\tau}$ and $\pi_{\sigma\upsilon} = \pi_{\upsilon\tau}^{-1} \circ \pi_{\sigma\tau}$. First note that π is structure-preserving.

Σ_1 is preserved upwards by π^{-1} (i.e., from $L_{s(\upsilon)}$ to $L_{s(\tau)}\{\alpha \cup p_\upsilon\}$). For the other direction, assume $L_{s(\tau)} \models \exists x\, \varphi(x, \vec{r})$, where φ is quantifier-free and $\vec{r} \in \operatorname{dom}\pi = L_{s(\tau)}\{\gamma_\upsilon \cup p_\tau\}$; we have to show $L_{s(\upsilon)} \models \exists x\, \varphi(x, \pi(\vec{r}))$. As before, fix $s_0 \mathrel{\widetilde{<}} s(\sigma)$ s. t. $\vec{r} \in L_{\pi_{\sigma\tau}(s_0)}\{\gamma_\upsilon \cup p_\tau\}$ and $w \in L_{\pi_{\sigma\tau}(s_0)}\{\gamma_\tau \cup p_\tau\}$ where w is the least witness for $\exists x\, \varphi(x, \vec{r})$. Our aim is to show that γ_τ can be replaced by γ_υ in the latter hull.

Let $\pi_1\colon L_{s_0}\{\gamma_\sigma \cup p_\sigma\} \cong L_{\bar{s}} = L_{\bar{s}}\{\gamma_\sigma \cup \bar{p}\}$ where $\bar{p} = \pi_1(p_\sigma)$. As above using type preservation, we shift π_1 to the γ_τ-level, let's call the resulting map $\pi_2\colon L_{\pi_{\sigma\tau}(s_0)}\{\gamma_\tau \cup p_\tau\} \cong L_{\pi_{\sigma\tau}(\bar{s})}\{\gamma_\tau \cup \pi_{\sigma\tau}(\bar{p})\}$. Then we have $\pi_2(\vec{r}) \in L_{\pi_{\sigma\tau}(\bar{s})}\{\gamma_\upsilon \cup \pi_{\sigma\tau}(\bar{p})\}$ and $\pi_2(w) \in L_{\pi_{\sigma\tau}(\bar{s})}\{\gamma_\tau \cup \pi_{\sigma\tau}(\bar{p})\}$: $L_{\pi_{\sigma\tau}(\bar{s})} \models \varphi(\pi_2(w), \pi_2(\vec{r}))$

Further, also as above, there is a $\bar{\sigma} \prec \sigma$ s. t. $L_{\bar{s}} \in L_{\bar{\sigma}}$ with $\bar{\sigma} \multimap \bar{\upsilon} \multimap \bar{\tau} := \pi_{\sigma\tau}(\bar{\sigma})$ and $\pi_2(\vec{r}), \pi_{\sigma\tau}(\bar{s}), \pi_{\sigma\tau}(\bar{p}) \in \operatorname{range}\pi_{\bar{\upsilon}\bar{\tau}}$. Therefore, $\pi_2(w) \in \operatorname{range}\pi_{\bar{\upsilon}\bar{\tau}}$ and hence by $\pi_{\bar{\upsilon}\bar{\tau}}$ being a morass map, we can replace γ_τ by γ_υ in "$\pi_2(w) \in L_{\pi_{\sigma\tau}(\bar{s})}\{\gamma_\tau \cup \pi_{\sigma\tau}(\bar{p})\}$". Applying π_2^{-1} we get $w \in \operatorname{range}\pi_{\upsilon\tau}$. This proves Σ_1-preservation. $\dashv$

Gap-2

Definition 15 (Gap-2-Morass) An $(\omega_1, 2)$-*morass* is a structure of the form $(\langle S_\gamma | \gamma \in S^- \rangle, \langle \pi_{\sigma\tau} | \sigma \multimap \tau \rangle)$ with

$$
\begin{array}{lcl}
S^i & \subset & \omega_3 \text{ for } i \in 3, \\
\multimap & \subset & (S^1 \times S^1) \cup (S^2 \times S^2), \\
\prec & \subset & (S^1 \times S^1) \cup (S^2 \times S^2), \\
S^- & := & S^0 \cup S^1, \\
S^+ & := & S^1 \cup S^2, \\
\gamma_\sigma & \in & S^i \text{ for } \sigma \in S^{i+1}, \\
S_\gamma & := & \left\{ \nu \in S^+ \,\middle|\, \gamma_\nu = \gamma \right\} \text{ for } \gamma \in S^-, \\
\nu_\gamma & := & \gamma \cup \max S_\gamma \text{ for } \gamma \in S^-, \text{ and} \\
d_\sigma & := & \begin{cases} \nu_\sigma + 1 & \text{for } \sigma \in S^1 \\ \sigma + 1 & \text{for } \sigma \in S^2 \end{cases}
\end{array}
$$

and the following properties:

i) S^0 is club in ω_1, $S^1 \setminus \omega_1$ club in ω_2, $S^2 \setminus \omega_2$ club in ω_3
For $\gamma \in S^-$, S_γ closed.
If $\gamma < \gamma'$ in S^i $(i \in 2)$ then $\nu_\gamma < \gamma'$
If $\sigma \multimap \tau \multimap \upsilon$ then $\pi_{\sigma\upsilon} = \pi_{\tau\upsilon} \circ \pi_{\sigma\tau}$.
$\multimap$ is a tree-ordering.

ii) $\pi_{\sigma\tau} \colon d_\sigma \xrightarrow{\text{o. p.}} d_\tau$
$\pi_{\sigma\tau} \restriction \gamma_\sigma = \mathrm{id} \restriction \gamma_\sigma$, $\pi_{\sigma\tau}(\gamma_\sigma) = \gamma_\tau$, and $\pi_{\sigma\tau}(\sigma) = \tau$
If $\sigma \multimap \tau$ in S^1 with $\sigma < \nu_\sigma$ then $\pi_{\sigma\tau}(\nu_\sigma) = \nu_\tau$.
If $\sigma \multimap \tau$ then $\gamma_\sigma < \gamma_\tau$ and $\tau < \omega_2$ if $\tau \in S^1$.
If $\sigma \multimap \tau$ then for all $\nu \preceq \sigma$, ν is $\prec$-minimal, successor, limit iff $\pi_{\sigma\tau}(\nu)$ is $\prec$-minimal, successor, limit, respectively.
If $\sigma \multimap \tau$ in S^1 with $\sigma < \nu_\sigma$ then for all $\nu \preceq \nu_\sigma$, ν is $\prec$-minimal, successor, limit iff $\pi_{\sigma\tau}(\nu)$ is $\prec$-minimal, successor, limit, respectively.
In the successor cases also the predecessor is preserved.
If $\sigma \multimap \tau$ then $\pi_{\sigma\tau}^{-1}\left[S_{\gamma_\tau} \cap (\tau + 1)\right] = S_{\gamma_\sigma} \cap (\sigma + 1)$

iii) If $\bar\sigma \prec \sigma$ and $\bar\tau = \pi_{\sigma\tau}(\bar\sigma)$ then $\bar\sigma \multimap \bar\tau$ and $\pi_{\bar\sigma\bar\tau} = \pi_{\sigma\tau} \restriction d_{\bar\sigma}$.
If $\tau \in S^+$ then:
$\{\gamma_\sigma \mid \sigma \multimap \tau\}$ is closed in γ_τ,
if τ is not $\prec$-maximal then $\{\gamma_\sigma \mid \sigma \multimap \tau\}$ is unbounded in γ_τ, and

if $\{\gamma_\sigma \mid \sigma \dashv\!3\, \tau\}$ is unbounded in γ_τ then $\tau = \bigcup_{\sigma \dashv 3 \tau} \operatorname{range} \pi_{\sigma\tau} \restriction \sigma$ and, for $\tau \in S^1$, $\nu_\tau = \bigcup_{\sigma \dashv 3 \tau} \operatorname{range} \pi_{\sigma\tau} \restriction \nu_\sigma$.

iv) If $\sigma \dashv\!3\, \tau$ in S^1 then: $\pi_{\sigma\tau}^{-1}[S_\tau] = S_\sigma$,
if σ is a $\prec$-limit and $\lambda := \sup \operatorname{range}(\pi_{\sigma\tau} \restriction \sigma) < \tau$ then $\sigma \dashv\!3\, \lambda$ and:
for $\sigma \dashv\!3\, \tau$ in S^1: $\{\nu \mid \nu \dashv\!3\, \nu_\lambda\} = \{\nu < \nu_\lambda \mid \nu \dashv\!3\, \nu_\tau\}$ and $\pi_{\sigma\tau} \restriction \nu_\sigma = \pi_{\bar\mu\mu} \circ \pi_{\sigma\lambda} \restriction \nu_\sigma$ with $\bar\mu \dashv\!3\, \mu$ where $\bar\mu = \sup \operatorname{range} \pi_{\sigma\lambda} \restriction \nu_\sigma$ and $\mu = \sup \operatorname{range} \pi_{\sigma\tau} \restriction \nu_\sigma$;
for $\sigma \dashv\!3\, \tau$ in S^2: $\pi_{\sigma\tau} \restriction \sigma = \pi_{\sigma\lambda} \restriction \sigma$

v) If $\sigma \dashv\!3\, \tau$, σ a $\prec$-limit and $\sup \operatorname{range}(\pi_{\sigma\tau} \restriction \sigma) = \tau$ then for $\alpha \in S^-$, if $\forall \bar\sigma \prec \sigma\ \exists \bar\upsilon \in S_\alpha\ \bar\upsilon \dashv\!3\, \pi_{\sigma\tau}(\bar\sigma)$, then there exists $\upsilon \in S_\alpha$ s. t. $\upsilon \dashv\!3\, \tau$.

vi) If $\sigma \dashv\!3\, \tau \in S^1$ then $\sigma < \nu_\sigma$ iff $\tau < \nu_\tau$. In this case:
If $\bar\mu \dashv\!3\, \bar\nu < \nu_\sigma$, then $\pi_{\sigma\tau}(\bar\mu) =: \mu \dashv\!3\, \nu := \pi_{\sigma\tau}(\bar\nu)$.
If in addition $\bar\mu, \bar\nu \in S^2$, then $\pi_{\mu\nu} \circ \pi_{\sigma\tau} \restriction d_{\bar\mu} = \pi_{\sigma\tau} \circ \pi_{\bar\mu\bar\nu}$.

vii) If $\sigma \dashv\!3\, \tau$ in S^1 and $\sigma < \nu_\sigma$ then:
$\pi_{\sigma\tau}^{-1}[\{\gamma_{\bar\tau} \mid \bar\tau \dashv\!3\, \nu_\tau\}] = \{\gamma_{\bar\sigma} \mid \bar\sigma \dashv\!3\, \nu_\sigma\}$ and
$\pi_{\sigma\tau}^{-1}[\{\bar\tau \mid \bar\tau \dashv\!3\, \nu_\tau\}] = \{\bar\sigma \mid \bar\sigma \dashv\!3\, \nu_\sigma\}$
ν_σ is $\dashv\!3$-minimal, successor, limit iff ν_τ is.
If $\bar\nu \dashv\!3\, \nu_\sigma$ then $\nu := \pi_{\sigma\tau}(\bar\nu) \dashv\!3\, \nu_\tau$ with $\pi_{\nu\nu_\tau} \circ \pi_{\sigma\tau} \restriction d_{\bar\nu} = \pi_{\sigma\tau} \circ \pi_{\bar\nu\nu_\sigma}$ and $\pi_{\sigma\tau}(\sup \operatorname{range} \pi_{\bar\nu\nu_\sigma} \restriction \bar\nu) = \sup \operatorname{range} \pi_{\nu\nu_\tau} \restriction \nu$. Also, if $\bar\nu$ is the immediate $\dashv\!3$-predecessor of ν_σ, then ν is the immediate $\dashv\!3$-predecessor of ν_τ.

Remark 16 *The gap-1-morass axioms are included in those of gap-2:*

	i)	*ii)*	*iii)*	*iv)*	*v)*	*≥vi)*
M0	×					
M1		×				
M2			×			
M3			×			
M4			×			
M5			×			
M6				×		
M7					×	

Definition 17 For $i \in 3$ define

$$S^i := \{\nu \in \omega_3 \mid \lim \nu \wedge L_\nu \models \text{“}\aleph_i \text{ is the largest cardinal”} \wedge \wedge\ \nu = \bigcup \{\mu < \nu \mid L_\mu \models ZF^-\}\} \ (\textit{morass points}) \text{ and}$$

$\gamma_\nu :=$ the largest cardinal in L_ν for $\nu \in S^+$ (the level of ν).

For $\mu, \nu \in S^+$ we define $\mu \prec \nu \Leftrightarrow \mu < \nu \wedge \gamma_\mu = \gamma_\nu$.

Note that $S_\gamma \subset S^{i+1}$ for $\gamma \in S^i$. Furthermore, μ, ν both in S^1 or both in S^2.

Let $s(\nu)$ be the $\tilde{\leq}$-least location s s. t. there is a $p \in^{<\omega} L_{\alpha(s)}$ with $L_s\{\gamma_\nu \cup p\} \cap \nu$ cofinal in ν; in this case let p_ν be the $<^*$-least such.

For σ, τ both in S^1 or both in S^2, let $\sigma \multimap\!\!3\, \tau$ iff exists $\pi \colon L_{s(\sigma)} \to L_{s(\tau)}$ s. t.

i) π Σ_1-preserving (in $\mathcal{L}$),

ii) $\pi \restriction \gamma_\sigma = \mathrm{id}$, $\pi(\gamma_\sigma) = \gamma_\tau > \gamma_\sigma$, $\pi(\sigma) = \tau$, $p_\tau \in \operatorname{range} \pi$

iii) If τ' is the immediate $\prec$-predecessor of τ, then $\tau' \in \operatorname{range} \pi$

iv) For $\sigma \in S^1$ with $\sigma < \nu_\sigma$:

(a) $\nu_\tau, s(\nu_\tau), p_{\nu_\tau} \in \operatorname{range} \pi$ (as usual, $\alpha(s(\tau))$ is considered an element of $\operatorname{range} \pi$),

(b) if ν is the immediate $\prec$-predecessor of ν_τ then $\nu \in \operatorname{range} \pi$,

(c) if $s, p \in \operatorname{range} \pi$ with $s \mathrel{\tilde{<}} s(\nu_\tau)$ and $p \in L_s$, then there exists $\beta \in \operatorname{range} \pi$ with $\beta < \nu_\tau$ and $L_s\{\tau \cup p\} \cap \nu_\tau \subset \beta$,

(d) if ν_τ is a $\multimap\!\!3$-successor with immediate predecessor ν then γ_ν, $\sup\left(L_{s(\nu_\tau)}\{\gamma_\nu \cup p_{\nu_\tau}\} \cap \nu_\tau\right) \in \operatorname{range} \pi$,

(e) if ν_τ is a $\multimap\!\!3$-limit then $\{\gamma_\nu \mid \nu \multimap\!\!3\, \nu_\tau\} \cap \operatorname{range} \pi$ is cofinal in $\sup(\operatorname{range} \pi \cap \tau)$, and

(f) if σ is a $\prec$-limit, $\sup(\operatorname{range} \pi \cap \tau) = \lambda < \tau$ and $H = L_{\tilde{s}}\{\gamma_\tau \cup p_\tau\}$ with $\tilde{s} = \tilde{\leq}\text{-lub}\left\{\pi(t) \,\middle|\, t \mathrel{\tilde{<}} s(\sigma)\right\}$ then:

i. If $s, p \in H$ with $s \mathrel{\tilde{<}} s(\nu_\tau)$ and $p \in L_s$, then there exists $\beta \in H \cap \nu_\tau$ with $L_s\{\tau \cup p\} \cap \nu_\tau \subset \beta$.

ii. $H \cap \nu_\tau = L_{s(\nu_\tau)}\{\lambda \cup p_{\nu_\tau}\} \cap \nu_\tau$

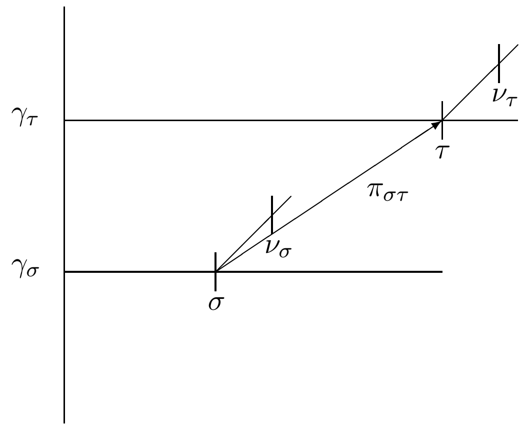
iii. $\sup(H \cap \nu_\tau) \in S_\tau$

iv. If ν_τ is a $\multimap\!\!3$-limit, then for all $\nu \multimap\!\!3\, \nu_\tau$ with $\gamma_\nu \in \operatorname{range} \pi$, $\sup(L_{s(\nu_\tau)}\{\gamma_\nu \cup p_{\nu_\tau}\} \cap \nu_\tau) \in \operatorname{range} \pi$.

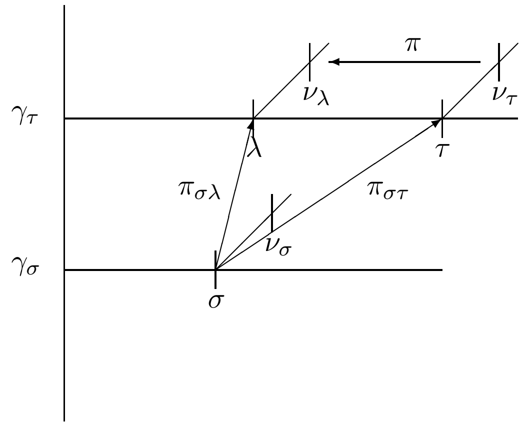
Remark 18 *Compared with the definition for the gap-1-maps, iv) is new. Properties (a) and (b) are analogous to ii) and iii). (c) is needed to show that $s(\nu_\sigma)$ and $s(\nu_\tau)$ match (lemma 23). (d) is used to show that the immediate $\multimap\!\!3$-predecessor of ν_τ is preserved (lemma 28). (e) implies that ν_σ is a $\multimap\!\!3$-limit if ν_τ is a $\multimap\!\!3$-limit. (f) will make sure that we get (M6). In particular*

(f)i and (f)ii show that $s(\nu_\lambda)$ is correct. By (f)iii ν_τ and ν_λ correspond. Finally, (f)iv is needed to show that suprema of morass-maps into ν_σ, ν_τ respectively, match (lemma 29).

Remark 19 *The typical "gap-2-map" will look like this:*



"(M6)" (i.e., iv)) might look like this:



The following is as for gap-1.

Lemma 20

i) $\nu \subset L_{s(\nu)}\{\gamma_\nu \cup p_\nu\}$

ii) $L_{s(\nu)}\{\gamma_\nu \cup p_\nu\} = L_{s(\nu)}$

iii) If $\pi \colon L_{s(\sigma)} \to L_{s(\tau)}$ *exists then* $\pi(p_\sigma) = p_\tau$.

iv) The map $\pi\colon L_{s(\sigma)} \to L_{s(\tau)}$, if exists, is uniquely determined. Furthermore, $L_{s(\sigma)} \cong L_{s(\tau)}\{\gamma_\sigma \cup p_\tau\} = \operatorname{range}\pi$.

v) If $\pi_{\nu\tau}$ is a morass map and τ' is the immediate $\prec$-predecessor of τ in S_{α_τ}, then $\pi_{\nu\tau}^{-1}(\tau')$ is the immediate predecessor of ν in S_{α_ν}. $\dashv$

Definition 21 (morass map) For $\sigma \multimap 3\ \tau$, let $\pi_{\sigma\tau}$ be the unique map from the previous lemma. The actual morass map to satisfy the morass axioms will be $\pi_{\sigma\tau}{\restriction}\max(d_\sigma)\cup\{(\max(d_\sigma), \max(d_\tau))\}$ (note that for $\max(d_\sigma)\in \operatorname{dom}\pi_{\sigma\tau}$ we have $\pi_{\sigma\tau}(\max(d_\sigma)) = \max(d_\tau)$), but we will write $\pi_{\sigma\tau}$ for both maps and work with the underlying map only.

Remark 22 *Let $\sigma \in S^1$. Then there are three possibilities for ν_σ:*

i) $(\sigma^+)^{L_{s(\sigma)}}$ exists, then ν_σ is that cardinal successor.

ii) There are infinitely many μ s. t. $L_\mu \models ZF^- \wedge \aleph_2 = \sigma$, then ν_σ is the largest limit of those. Note: The first case actually is a subcase of this.

iii) Else $\nu_\sigma = \sigma$.

Lemma 23 *Let $\sigma \in S^1$ with $\sigma < \nu_\sigma$ and let $\pi_{\sigma\tau}$ satisfy the definition of a morass map with the possible exception of iv)(d) to (f). Then:*

i) $\pi_{\sigma\tau}(\nu_\sigma) = \nu_\tau$.

ii) $\pi_{\sigma\tau}(s(\nu_\sigma)) = s(\nu_\tau)$.

iii) $\pi_{\sigma\tau}(p_{\nu_\sigma}) = p_{\nu_\tau}$.

iv) If ν is the immediate $\prec$-predecessor of ν_τ then $\pi_{\sigma\tau}^{-1}(\nu)$ is the immediate $\prec$-predecessor of ν_σ.

Proof For i) we only need to consider the second case of remark 22 (the third contradicts the assumption). First look at ν_σ as the limit of the μ's. Those are preserved, hence $\pi_{\sigma\tau}(\nu_\sigma)$ also is a limit of such μ's. Assume it is not the biggest such. Then ν_τ is the largest limit of μ's and greater than $\pi_{\sigma\tau}(\nu_\sigma)$. This can be expressed by a Σ_1-formula. But then also $\pi_{\sigma\tau}^{-1}(\nu_\tau)$ is of this form, contradicting the maximality of ν_σ. For the other direction, assume that $\pi_{\sigma\tau}^{-1}(\nu_\tau) < \nu_\sigma$. As before we would get that $\pi_{\sigma\tau}(\nu_\sigma)$ is a limit of μ's, but now bigger than ν_τ, contradicting the maximality of ν_τ.

For ii) first assume that $\pi_{\sigma\tau}(s(\nu_\sigma)) \mathrel{\widetilde{>}} s(\nu_\tau)$. By definition of $s(\nu_\sigma) \mathrel{\widetilde{>}} \pi_{\sigma\tau}^{-1}(s(\nu_\tau))$ there is a $\beta < \nu_\sigma$ s. t. $L_{\pi_{\sigma\tau}^{-1}(s(\nu_\tau))}\left\{\sigma \cup \pi_{\sigma\tau}^{-1}(p_{\nu_\tau})\right\} \cap \nu_\sigma$ is bounded

by β. But then also $L_{s(\nu_\tau)}\{\tau \cup p_{\nu_\tau}\} \cap \nu_\tau$ is bounded, namely by $\pi_{\sigma\tau}(\beta)$ — contradicting the definition of $s(\nu_\tau)$, p_{ν_τ}. For the other direction assume $\pi_{\sigma\tau}(s(\nu_\sigma)) \tilde{<} s(\nu_\tau)$. By the definition of $s(\nu_\tau)$, $L_{\pi_{\sigma\tau}(s(\nu_\sigma))}\{\tau \cup \pi_{\sigma\tau}(p_{\nu_\sigma})\} \cap \nu_\tau$ is bounded below ν_τ; by iv)(c) of the definition of a morass map, there exists a bound in range $\pi_{\sigma\tau}$, β say. But then also $L_{s(\nu_\sigma)}\{\sigma \cup p_{\nu_\sigma}\} \cap \nu_\sigma$ is bounded by $\pi_{\sigma\tau}^{-1}(\beta) < \nu_\sigma$ — contradicting the definition of $s(\nu_\sigma)$, p_{ν_σ}.

For iii) note $p_{\nu_\tau} \in \operatorname{range} \pi_{\sigma\tau}$. By definition of p_{ν_τ}, $\pi_{\sigma\tau}(p_{\nu_\sigma}) \in L_{s(\nu_\tau)}\{\tau \cup p_{\nu_\tau}\}$. Therefore, $p_{\nu_\sigma} \in L_{s(\nu_\sigma)}\left\{\sigma \cup \pi_{\sigma\tau}^{-1}(p_{\nu_\tau})\right\}$ and hence $\pi_{\sigma\tau}^{-1}(p_{\nu_\tau})$ collapses ν_σ, so $p_{\nu_\sigma} \leq^* \pi_{\sigma\tau}^{-1}(p_{\nu_\tau})$. On the other hand, $\pi_{\sigma\tau}^{-1}(p_{\nu_\tau}) \in L_{s(\nu_\sigma)}\{\sigma \cup p_{\nu_\sigma}\} = L_{s(\nu_\sigma)}$. Therefore, $p_{\nu_\tau} \in L_{s(\nu_\tau)}\{\tau \cup \pi_{\sigma\tau}(p_{\nu_\sigma})\}$ and $\pi_{\sigma\tau}(p_{\nu_\sigma})$ collapses ν_τ, hence $p_{\nu_\tau} \leq^* \pi_{\sigma\tau}(p_{\nu_\sigma})$. So $\pi_{\sigma\tau}(p_{\nu_\sigma})$ and p_{ν_τ} must be equal.

For iv) assume that there is $\nu' \prec \nu_\sigma$ with $\nu' > \pi_{\sigma\tau}^{-1}(\nu)$. But then $\nu < \pi_{\sigma\tau}(\nu') \prec \nu_\tau$, contradiction. $\dashv$

The first part of the proof actually shows:

Corollary 24 *Let $\sigma \in S^1$ and let $\pi_{\sigma\tau}$ satisfy the definition of a morass map with the possible exception of iv)(b) to (f). Then $\sigma < \nu_\sigma$ iff $\tau < \nu_\tau$.* $\dashv$

The second part of the proof actually shows:

Corollary 25 *Let $\sigma \in S^1$ and let $\pi_{\sigma\tau}$ satisfy the definition of a morass map with the possible exception of iv)(d) to (f). If $s, p, \beta \in \operatorname{range} \pi_{\sigma\tau}$ with $s \tilde{<} s(\nu_\tau)$, $p \in L_s$ and $L_s\{\tau \cup p\} \cap \nu_\tau \subset \beta < \nu_\tau$, then $L_{\pi_{\sigma\tau}^{-1}(s)}\left\{\sigma \cup \pi_{\sigma\tau}^{-1}(p)\right\} \cap \nu_\sigma \subset \pi_{\sigma\tau}^{-1}(\beta)$. Also, if $\bar{s} \tilde{<} s(\nu_\sigma)$, $\bar{p} \in L_{\bar{s}}$ and $L_{\bar{s}}\{\sigma \cup \bar{p}\} \cap \nu_\sigma \subset \bar{\beta} < \nu_\sigma$, then $L_{\pi_{\sigma\tau}(\bar{s})}\{\tau \cup \pi_{\sigma\tau}(\bar{p})\} \cap \nu_\tau \subset \pi_{\sigma\tau}(\bar{\beta})$.* $\dashv$

Lemma 26 *Let $\sigma \in S^1$ with $\sigma < \nu_\sigma$ and let $\pi_{\sigma\tau}$ satisfy the definition of a morass map with the possible exception of iv)(d) to (f). $\sigma' \prec \sigma$ is the level of a $\dashv$3-predecessor $\bar{\nu}$ of ν_σ iff $\pi_{\sigma\tau}(\sigma') \prec \tau$ is the level of a $\dashv$3-predecessor ν of ν_τ. In this case $\pi_{\sigma\tau}(\bar{\nu}) = \nu$.*

Proof First assume that $\sigma' \prec \sigma$ is the level of a $\dashv$3-predecessor $\bar{\nu}$ of ν_σ:

$$\pi_{\bar{\nu}\nu_\sigma} \colon L_{s(\bar{\nu})} = L_{s(\bar{\nu})}\left\{\sigma' \cup p_{\bar{\nu}}\right\} \cong L_{s(\nu_\sigma)}\left\{\sigma' \cup p_{\nu_\sigma}\right\}$$

By type preservation and elementarity of $\pi_{\sigma\tau} \restriction L_{s(\nu)}$ we get:

$$\pi \colon L_{\pi_{\sigma\tau}(s(\bar{\nu}))} = L_{\pi_{\sigma\tau}(s(\bar{\nu}))}\left\{\tau' \cup \pi_{\sigma\tau}(p_{\bar{\nu}})\right\} \cong L_{s(\nu_\tau)}\left\{\tau' \cup p_{\nu_\tau}\right\}$$

where $\tau' = \pi_{\sigma\tau}(\sigma')$. π is a morass map: $\pi \restriction \tau' = \mathrm{id} \restriction \tau'$; $L_{s(\nu_\sigma)}\{\tau' \cup p_{\nu_\tau}\} \cap \tau = \tau'$ by Σ_1-preservation, hence $\pi(\tau') = \tau$; $\pi(\pi_{\sigma\tau}(\bar{\nu})) = \nu_\tau$ by structure-preservation for $\pi_{\bar{\nu}\nu_\sigma}$, $\pi_{\sigma\tau}$ and π; $p_{\nu_\tau} \in \mathrm{range}\,\pi$; and the $\prec$-predecessor of ν_τ (if exists) is also in $\mathrm{range}\,\pi$. $\pi_{\sigma\tau}(s(\bar{\nu})) = s(\nu)$ by elementarity where $\nu = \pi_{\sigma\tau}(\bar{\nu})$.

Finally, we check that π is Σ_1-preserving. Assume that there is an $x \in L^*_{s(\nu_\tau)}\{\tau' \cup p_{\nu_\tau}\} \setminus L_{s(\nu_\tau)}\{\tau' \cup p_{\nu_\tau}\}$. Also we must have $x \in L_{s(\nu_\tau)}\{\tau \cup p_{\nu_\tau}\} = L_{s(\nu_\tau)}$. So there is a term t and some $\vec{\xi} < \tau$: $x = t\left(\vec{\xi}, p_{\nu_\tau}\right)$. Since $\tau, x, p_{\nu_\tau} \in L^*_{s(\nu_\tau)}\{\tau' \cup p_{\nu_\tau}\}$, the least such $\vec{\xi}$ is also in $L^*_{s(\nu_\tau)}\{\tau' \cup p_{\nu_\tau}\}$. Since by Σ_1-preservation, $L^*_{s(\nu_\tau)}\{\tau' \cup p_{\nu_\tau}\} \cap \tau = \tau'$ we get $\vec{\xi} < \tau'$. But then x is already in $L_{s(\nu_\tau)}\{\tau' \cup p_{\nu_\tau}\}$, hence both hulls agree, $\mathrm{range}\,\pi$ is Σ_1-closed and π Σ_1-preserving. Therefore π is the morass map $\pi_{\nu\nu_\tau}$.

For the other direction assume that $\tau' = \pi_{\sigma\tau}(\sigma') \prec \tau$ is the level of a $\prec$-predecessor ν of ν_τ: $\pi_{\nu\nu_\tau}\colon L_{s(\nu)} = L_{s(\nu)}\{\tau' \cup p_\nu\} \cong L_{s(\nu_\tau)}\{\tau' \cup p_{\nu_\tau}\}$. Define the corresponding collapse downstairs:

$$\pi\colon L_{\bar{s}} = L_{\bar{s}}\left\{\sigma' \cup \bar{p}\right\} \cong L_{s(\nu_\sigma)}\left\{\sigma' \cup p_{\nu_\sigma}\right\}$$

where $\bar{p} = \pi^{-1}(p_{\nu_\sigma})$. Let $\bar{\nu} = \pi^{-1}(\nu_\sigma)$. We want to show that π is the morass map $\pi_{\bar{\nu}\nu_\sigma}$, in which case — by the preceding considerations — we would have $\bar{\nu} = \pi^{-1}_{\sigma\tau}(\nu)$.

By definition, π is structure-preserving, $\pi \restriction \sigma' = \mathrm{id} \restriction \sigma'$, $\pi(\bar{\nu}) = \nu_\sigma$, and $p_{\nu_\sigma} \in \mathrm{range}\,\pi$. Next we show that $\pi(\sigma') = \sigma$, i.e., $L_{s(\nu_\sigma)}\{\sigma' \cup p_{\nu_\sigma}\} \cap \sigma = \sigma'$: Assume for contradiction that there is an $x \in L_{s(\nu_\sigma)}\{\sigma' \cup p_{\nu_\sigma}\} \cap \sigma \setminus \sigma'$. Applying $\pi_{\sigma\tau}$ we get $\pi_{\sigma\tau}(x) \in L_{s(\nu_\tau)}\{\tau' \cup p_{\nu_\tau}\} \cap \tau \setminus \tau'$, contradicting that $\pi_{\nu\nu_\tau}$ is a morass map.

Note that by type preservation $\pi_{\sigma\tau}(\bar{s}) = s(\nu)$ and $\pi_{\sigma\tau}(\bar{p}) = p_\nu$. Therefore, by elementarity we have $\bar{s} = s(\bar{\nu})$.

To see that the $\prec$-predecessor of ν_σ, if exists (which we assume from now on), is in $\mathrm{range}\,\pi$, first observe that the $\prec$-predecessor of ν_τ is in $\mathrm{range}\,\pi_{\sigma\tau}$, call it μ_τ. $\mu_\sigma := \pi^{-1}_{\sigma\tau}(\mu_\tau)$ is the $\prec$-predecessor of ν_σ. Furthermore, $\mu_\tau \in \mathrm{range}\,\pi_{\nu\nu_\tau} = L_{s(\nu_\tau)}\{\tau' \cup p_{\nu_\tau}\}$. By Σ_1-preservation for $\pi_{\sigma\tau}$ we get $\mu_\sigma \in L_{s(\nu_\sigma)}\{\sigma' \cup p_{\nu_\sigma}\} = \mathrm{range}\,\pi$.

To see that π is Σ_1-preserving let φ be a QFF and assume $L_{s(\nu_\sigma)} \models \exists v \varphi(v, \vec{x})$ where $\vec{x} \in \mathrm{range}\,\pi$. We have to find a $v \in L_{s(\bar{\nu})}$ s. t. $L_{s(\bar{\nu})} \models \varphi\left(v, \pi^{-1}(\vec{x})\right)$.

Using $\pi_{\sigma\tau}$ we have $L_{s(\nu_\tau)} \models \exists v\, \varphi\,(v, \pi_{\sigma\tau}(\vec{x}))$ where $\pi_{\sigma\tau}(\vec{x}) \in \operatorname{range} \pi_{\nu\nu_\tau}$. A witness w for the existential quantifier is also in $\operatorname{range} \pi_{\nu\nu_\tau}$ and hence — by Σ_1-preservation of $\pi_{\sigma\tau}$ — $\pi_{\sigma\tau}^{-1}(w) \in \operatorname{range} \pi$ as required. $\dashv$

Corollary 27 *Let $\sigma \in S^1$ with $\sigma < \nu_\sigma$ and let $\pi_{\sigma\tau}$ satisfy the definition of a morass map with the possible exception of iv)(d) to (f). Then the following are equivalent:*

i) iv)(e) of the morass map definition: if ν_τ is a $\lhd$-limit then $\{\gamma_\nu \mid \nu \lhd \nu_\tau\} \cap \operatorname{range} \pi_{\sigma\tau}$ is cofinal in $\sup(\operatorname{range} \pi_{\sigma\tau} \cap \tau)$

ii) ν_σ is a $\lhd$-limit iff ν_τ is a $\lhd$-limit. $\dashv$

Lemma 28 *Let $\sigma \in S^1$ with $\sigma < \nu_\sigma$ and let $\pi_{\sigma\tau}$ satisfy the definition of a morass map with the possible exception of iv)(e) and (f). If ν is the immediate $\lhd$-predecessor of ν_τ and $\beta = \sup\left(L_{s(\nu_\tau)}\,\{\gamma_\nu \cup p_{\nu_\tau}\} \cap \nu_\tau\right)$, then $\bar{\nu} = \pi_{\sigma\tau}^{-1}(\nu)$ is the immediate $\lhd$-predecessor of ν_σ and for $\bar{\beta} = \pi_{\sigma\tau}^{-1}(\beta)$ we have $\bar{\beta} = \sup\left(L_{s(\nu_\sigma)}\,\{\gamma_{\bar{\nu}} \cup p_{\nu_\sigma}\} \cap \nu_\sigma\right)$.*

Proof First note that $\nu \in \operatorname{range} \pi_{\sigma\tau}$ (by lemma 26). If there were a bigger $\lhd$-predecessor of ν_σ, its image under $\pi_{\sigma\tau}$ would be a predecessor of ν_τ, contradicting our assumption. Clearly $\bar{\beta}$ is an upper bound of $L_{s(\nu_\sigma)}\,\{\gamma_{\bar{\nu}} \cup p_{\nu_\sigma}\} \cap \nu_\sigma$, else by structure-preservation β would not be a bound. Assume there is a smaller bound $\bar{\beta}'$ and let $\beta' = \pi_{\sigma\tau}(\bar{\beta}')$; then a Σ_1-statement says that there exists a witness that β' is not a bound, which is preserved downwards, hence $\bar{\beta}'$ cannot be a bound. $\dashv$

Lemma 29 *If $\sigma \lhd \tau$ in S^1, $\sigma < \nu_\sigma$, $\bar{\nu} \lhd \nu_\sigma$, and $\bar{\beta} = \sup(\operatorname{range} \pi_{\bar{\nu}\nu_\sigma} \cap \nu_\sigma)$, then $\pi_{\sigma\tau}(\bar{\beta}) = \beta$, where $\beta = \sup(\operatorname{range} \pi_{\nu\nu_\tau} \cap \nu_\tau)$ and $\nu = \pi_{\sigma\tau}(\bar{\nu})$.*

Proof It is enough to show that $\beta \in \operatorname{range} \pi_{\sigma\tau}$, since then by Σ_1-preservation for $\pi_{\sigma\tau}$, its preimage must be $\bar{\beta}$.

Suppose $\beta \notin \operatorname{range} \pi_{\sigma\tau}$, in particular $\beta < \nu_\tau$. Either ν_τ is a $\lhd$-limit or if ν' is the immediate $\lhd$-predecessor of ν_τ, then $\beta < \sup(\operatorname{range} \pi_{\nu'\nu_\tau} \cap \nu_\tau)$, since otherwise iv)(d) implies that $\beta \in \operatorname{range} \pi_{\sigma\tau}$.

There exists $\nu_0 \lhd \nu_\tau$ s. t. $\nu_0 > \nu$, $\nu_0 = \pi_{\sigma\tau}(\bar{\nu}_0)$ and $\beta < \sup(\operatorname{range} \pi_{\nu_0\nu_\tau} \cap \nu_\tau)$: For, if ν_τ is a $\lhd$-successor, ν' is as required; if ν_τ is a $\lhd$-limit, then $\operatorname{range} \pi_{\sigma\tau}$ cofinal in τ else by iv)(f)iv $\beta \in \operatorname{range} \pi_{\sigma\tau}$; but then by iv)(e) $\bigcup_{\nu_0 \lhd \nu_\tau} \operatorname{range} \pi_{\nu_0\nu_\tau} = L_{s(\nu_\tau)}\,\{\tau \cup p_{\nu_\tau}\} = L_{s(\nu_\tau)}$.

Now let $\beta_0 = \sup(\operatorname{range}\pi_{\nu\nu_0} \cap \nu_0)$, then $\beta_0 < \nu_0$ and $\beta \leq \pi_{\nu_0\nu_\tau}(\beta_0)$. But $\pi_{\nu_0\nu_\tau}\restriction\beta_0$ is fully elementary, hence $L_{s(\nu_0)} \models \sup(L_{s(\nu_0)}\{\gamma_\nu \cup p_{\nu_0}\} \cap \nu_0) = \beta_0$ is preserved to $L_{s(\nu_\tau)} \models \sup(L_{s(\nu_\tau)}\{\gamma_\nu \cup p_{\nu_\tau}\} \cap \nu_\tau) = \pi_{\nu_0\nu_\tau}(\beta_0)$. Therefore, $\beta = \pi_{\nu_0\nu_\tau}(\beta_0)$. Finally, since $\pi_{\bar{\nu}_0\nu_0} = \pi_{\sigma\tau} \restriction L_{s(\bar{\nu}_0)}$ is fully elementary, $\beta_0 \in \operatorname{range}\pi_{\bar{\nu}_0\nu_0}$. It follows that $\beta \in \operatorname{range}\pi_{\sigma\tau}$. $\dashv$

Theorem 30 $(\langle S_\gamma | \gamma \in S^- \rangle, \langle \pi_{\sigma\tau} | \sigma \multimap \tau \rangle)$ *as defined in definitions 17 and 21 is an* $(\omega_1, 2)$*-morass.*

Proof For **i)** the first two properties are clear. For the third, first note that $\gamma' \in S^i$ and $\nu_\gamma \in S^{i+1}$, hence $\gamma' \neq \nu_\gamma$. Now, $L_{\gamma'} \models \bar{\bar{\gamma}} \leq \aleph_i$, but $L_{\nu_\gamma} \models \gamma = \aleph_{i+1}$, hence $\nu_\gamma < \gamma'$.

For $\pi_{\sigma\upsilon} = \pi_{\tau\upsilon} \circ \pi_{\sigma\tau}$ if $\sigma \multimap \tau \multimap \upsilon$ the first three properties of the morass map definition are clear for $\pi_{\sigma\upsilon}$. For property iv) of the definition, (a) and (b) are clear using lemma 23, (c) follows from corollary 25, (d) is done by lemma 28, and for (e) use corollary 27.

Finally, to see (f) assume σ is a $\prec$-limit, $\sup(\operatorname{range}\pi_{\sigma\upsilon} \cap \upsilon) = \lambda < \upsilon$, $H = L_{\tilde{s}}\{\gamma_\upsilon \cup p_\upsilon\}$ with $\tilde{s} = \widetilde{\leq}\text{-lub}\{\pi_{\sigma\upsilon}(t) \mid t \,\widetilde{<}\, s(\sigma)\}$. We consider two cases: Firstly assume $\pi_{\sigma\tau}$ is cofinal into τ (and hence $\widetilde{\leq}\text{-lub}\{\pi_{\sigma\tau}(t) \mid t \,\widetilde{<}\, s(\sigma)\} = s(\tau)$), then $\tilde{s} = \widetilde{\leq}\text{-lub}\{\pi_{\tau\upsilon}(t) \mid t \,\widetilde{<}\, s(\sigma)\}$ and by (f) for $\pi_{\tau\upsilon}$ and lemma 29 we get (f) for $\pi_{\sigma\upsilon}$. Secondly assume that $\sup(\pi_{\sigma\tau} \cap \tau) = \bar{\lambda} < \tau$. Then $\pi_{\tau\upsilon} \restriction L_{s(\bar{\lambda})}$ is fully elementary, actually it is the map $\pi_{\bar{\lambda}\lambda_0}$ where $\lambda_0 := \pi_{\tau\upsilon}(\bar{\lambda})$. Observe that $L_{s(\bar{\lambda})} \models \sup(L_{s(\bar{\lambda})}\{\gamma_\sigma \cup p_{\bar{\lambda}}\} \cap \bar{\lambda}) = \bar{\lambda}$. But then by elementarity $L_{s(\lambda_0)} \models \sup(L_{s(\lambda_0)}\{\gamma_\sigma \cup p_{\lambda_0}\} \cap \lambda_0) = \lambda_0$ and hence $\pi_{\sigma\upsilon}$ cofinal into λ_0. Therefore, $\lambda_0 = \lambda$. Now by type preservation for $\pi_{\tau\upsilon}$ we have $L_{s(\bar{\lambda})} \cong L_{\tilde{\bar{s}}}\{\bar{\lambda} \cup p_\tau\}$ corresponding to $L_{s(\lambda)} \cong L_{\pi_{\tau\upsilon}(\tilde{\bar{s}})}\{\lambda \cup p_\upsilon\}$. By type preservation and elementarity this diagram commutes. But we already know that $L_{s(\lambda)} \cong L_{\tilde{s}}\{\lambda \cup p_\upsilon\}$. Since the locations in $\operatorname{range}\pi_{\sigma\upsilon}$ are cofinal in $\tilde{s}$ and smaller than $\pi_{\tau\upsilon}(\tilde{\bar{s}})$ we must have $\pi_{\tau\upsilon}(\tilde{\bar{s}}) = \tilde{s}$. Now using the commutative diagram with full elementarity and lemma 29, we get (f) for $\pi_{\sigma\upsilon}$.

To see that $\multimap$ is a tree-ordering we need to show for $\sigma \multimap \upsilon$, $\tau \multimap \upsilon$, and $\sigma < \tau$ that $\sigma \multimap \tau$, but $\pi_{\sigma\tau} = \pi_{\tau\upsilon}^{-1} \circ \pi_{\sigma\upsilon}$ is as required: The first three properties of the morass map definition are clear for $\pi_{\sigma\tau}$. Using the same arguments as before (a) to (d) hold since all the parameters match. For (e) again use corollary 27.

Finally, to see (f) assume σ is a $\prec$-limit, $\sup(\operatorname{range}\pi_{\sigma\tau}\cap\tau) = \lambda < \tau$, and $H = L_{\tilde{s}}\{\gamma_\tau \cup p_\tau\}$ with $\tilde{s} = \widetilde{\leq}\text{-lub}\left\{\pi_{\sigma\tau}(t) \,\middle|\, t \mathrel{\widetilde{<}} s(\sigma)\right\}$. It follows that $\sup(\operatorname{range}\pi_{\sigma\upsilon}\cap\upsilon) =: \tilde{\lambda} < \upsilon$. But now $\pi_{\tau\upsilon}\restriction L_{s(\lambda)}$ is fully elementary and hence — as before — we get $\pi_{\tau\upsilon}(\lambda) = \tilde{\lambda}$ and finally (f) for $\pi_{\sigma\tau}$.

For **ii)**, the first two properties are clear, the third is lemma 23; for $\sigma \dashv\!3\, \tau$, $\gamma_\sigma < \gamma_\tau$ by definition and $\tau < \omega_2$ for $\tau \in S^1$ since ω_2 is regular.

To see that $\prec$-minimality is preserved for the ν's, assume there is a smaller element in the corresponding $S_{\pi_{\sigma\tau}(\gamma_\nu)}$ (i.e., S_{γ_τ} or $S_{\gamma_{\nu_\tau}}$), this property would be preserved by elementarity. For $\nu \prec \sigma$ (ν_σ, respectively) the successor and limit properties are also preserved by elementarity. For σ, being a successor means that S_{γ_σ} is bounded below σ with supremum its predecessor, ν say. Hence $\pi_{\sigma\tau}(\nu)$ is a bound on the morass points $\prec \tau$ (recall from gap-1 that being a morass point is Σ_1, hence the existence of such is preserved). Therefore, $\pi_{\sigma\tau}(\nu)$ is the immediate predecessor of τ, as required. For the other direction (if τ is a successor) use its immediate predecessor. For ν_σ, being a successor means that S_σ is bounded, proceed as for σ. The other direction was shown in lemma 23. The preservation of limit follows from the preservation of successor.

We have $\pi_{\sigma\tau}^{-1}[S_{\gamma_\tau}\cap\tau] \supset S_{\gamma_\sigma}\cap\sigma$ because for every $\nu \in S_{\gamma_\sigma}\cap\sigma$ $\pi_{\sigma\tau}(\nu) \in S_{\gamma_\tau}$ and $\pi_{\sigma\tau}(\nu) < \pi_{\sigma\tau}(\sigma) = \tau$. On the other hand, $\pi_{\sigma\tau}^{-1}[S_{\gamma_\tau}\cap\tau] \subset S_{\gamma_\sigma}\cap\sigma$ since every morass point in the range ($< \tau$) must be the image of a morass point by elementarity.

In **iii)**, the first property is like (M2) in the gap-1 case. Assume $\bar{\sigma} \prec \sigma$ and $\bar{\tau} = \pi_{\sigma\tau}(\bar{\sigma})$. Then $\bar{\tau} \prec \tau$ by the previous considerations and $\pi_{\sigma\tau}\restriction L_{s(\bar{\sigma})} = \pi_{\bar{\sigma}\bar{\tau}}$ by elementarity as required.

Assume $\tau \in S^+$ from now on. Now, the second property is like (M3) in the gap-1 case. We need to check that the constructed map $\pi_{\bar{\tau}\tau}$ also satisfies property iv) of the gap-2-morass definition. So assume that $\sigma \in S^1$ with $\sigma < \nu_\sigma$. Since $\operatorname{range}\pi_{\bar{\tau}\tau}$ is the union of ranges of morass maps, properties (a), (b), (d), and (e) hold. (c) holds for members of the union which are big enough to capture s, p. Similarly for (f).

The third property is like (M4) in the gap-1 case. The same proof works since the additional needed parameters are definable; by full elementarity the resulting map then is a morass map.

The first part of the fourth property is exactly like (M5) in the gap-1 case, the second part is true by the same argument.

iv) The first property is vacuous for $\sigma = \nu_\sigma$, else: $\pi_{\sigma\tau}(\nu_\sigma) = \nu_\tau$ by lemma 23. The morass points $\prec \nu_\sigma$ (ν_τ, respectively) are preserved by elementarity.

The second property is like (M6) in the gap-1 case. Only condition iv) for $\pi_{\sigma\lambda}$ is new in case $\sigma < \nu_\sigma$ ($\sigma \in S^1$). Recall $\tilde{s} = \widetilde{\leq}$-lub $\{\pi_{\sigma\tau}(t) \mid t \mathrel{\widetilde{<}} s(\sigma)\}$, $H := L_{\tilde{s}}\{\gamma_\tau \cup p_\tau\}$, $\pi \colon H \cong L_{s(\lambda)}$, $\pi \restriction \lambda = \mathrm{id}$, $\pi(\tau) = \lambda$, $\pi(p_\tau) = p_\lambda$, and $\pi_{\sigma\lambda} = \pi \circ \pi_{\sigma\tau}$ in the gap-1 sense.

First an observation about $\tilde{s}$: We know that $\tilde{s} \mathrel{\widetilde{<}} s(\tau)$ else $\pi_{\sigma\tau}$ were cofinal into τ. Assume $\alpha(\tilde{s}) < \alpha(s(\tau))$ and observe that this implies that $s(\tau)$ is of the form $(\beta, \varphi_0, \emptyset)$, for if not $\tilde{s} \mathrel{\widetilde{<}} (\alpha(s(\tau)), \varphi_0, \emptyset) \in \operatorname{range} \pi_{\sigma\tau}$. If $\beta = \beta' + 1$ then $\tilde{s} = \lim_{n<\omega}(\beta', \varphi_n, \emptyset) = s(\tau)$ contradicting our assumption. If β is a limit then $L_{s(\tau)} = L_{s(\tau)}\{\gamma_\tau \cup p_\tau\} = L\{\gamma_\tau \cup p_\tau\} = L_{s'}\{\gamma_\tau \cup p_\tau\}$ for any s' of the form $(\gamma, \varphi_0, \emptyset)$ with $\gamma > \max p_\tau$, contradicting the minimality of $s(\tau)$. Therefore, we have $\alpha(\tilde{s}) = \alpha(s(\tau))$.

First we check iv)(a): For $\nu_\lambda \in \operatorname{range} \pi_{\sigma\lambda}$, assume this is not the case. First assume that $\pi(\nu_\tau) > \nu_\lambda$. $\pi(\nu_\tau) \in S_\lambda$ since by iv)(f)iii for $\pi_{\sigma\tau}$ there are cofinally many ZF^--models which think that $\aleph_2$ is the largest cardinal in $H \cap \nu_\tau$ and those are preserved by elementarity for suitable restrictions of π. If on the other hand $\pi(\nu_\tau) < \nu_\lambda$, observe that $\pi^{-1}(\nu_\lambda)$ is a limit of ZF^--models which think that $\tau = \aleph_2$ is the largest cardinal (note that $\nu_\lambda < \alpha(s_\lambda)$, else it would be in range $\pi_{\sigma\lambda}$), contradicting the definition of ν_τ. Therefore, $\nu_\lambda = \pi(\nu_\tau) \in \operatorname{range} \pi_{\sigma\lambda}$.

To see that $s(\nu_\lambda) \in \operatorname{range} \pi_{\sigma\lambda}$, assume this is not the case, i.e., $s(\nu_\lambda) \mathrel{\widetilde{<}} s(\lambda)$ and $s(\nu_\lambda) \neq \pi_{\sigma\lambda}(s(\nu_\sigma)) = \pi(s(\nu_\tau))$. Assume first that $s(\nu_\lambda) \mathrel{\widetilde{<}} \pi(s(\nu_\tau))$ or $\pi^{-1}(s(\nu_\lambda)) \mathrel{\widetilde{<}} s(\nu_\tau)$. Hence $L_{\pi^{-1}(s(\nu_\lambda))}\{\tau \cup \pi^{-1}(p_{\nu_\lambda})\} \cap \nu_\tau$ is bounded; by iv)(f)i of the morass definition for $\pi_{\sigma\tau}$ there is a bound in $\operatorname{dom} \pi$, β say. But then $L_{s(\nu_\lambda)}\{\lambda \cup p_{\nu_\lambda}\} \cap \nu_\lambda$ is bounded by $\pi(\beta)$, contradicting the definition of $s(\nu_\lambda), p_{\nu_\lambda}$. If on the other hand $s(\nu_\lambda) \mathrel{\widetilde{>}} \pi(s(\nu_\tau))$ then $L_{\pi(s(\nu_\tau))}\{\lambda \cup \pi(p_{\nu_\tau})\} \cap \nu_\lambda$ is bounded, by $\beta < \nu_\lambda$ say. But $\pi^{-1}(\beta) \in H$ and $\pi^{-1}(\beta) < \nu_\tau$, so by iv)(f)ii $\pi^{-1}(\beta)$ cannot be a bound on $L_{s(\nu_\tau)}\{\lambda \cup p_{\nu_\tau}\} \cap \nu_\tau$.

For $p_{\nu_\lambda} \in \operatorname{range} \pi_{\sigma\lambda}$ note that by the previous argument $L_{s(\nu_\lambda)}\{\lambda \cup \pi(p_{\nu_\tau})\}$ cofinalizes ν_λ. Therefore, $p_{\nu_\lambda} \leq^* \pi(p_{\nu_\tau})$. Assume p_{ν_λ} is strictly smaller. We have $\pi(p_{\nu_\tau}) \in L_{s(\nu_\lambda)}\{\lambda \cup p_{\nu_\lambda}\}$, hence $p_{\nu_\tau} \in L_{s(\nu_\tau)}\{\tau \cup \pi^{-1}(p_{\nu_\lambda})\}$, contradicting the minimality of p_{ν_τ}. This concludes the proof of iv)(a) for $\pi_{\sigma\lambda}$.

For (b) assume that ν is the immediate $\prec$-predecessor of ν_λ. Then $\pi^{-1}(\nu) \prec \nu_\tau$ (by full elementarity of a suitable restriction of π). Assume there is a ν' s. t. $\pi^{-1}(\nu) \prec \nu' \prec \nu_\tau$. The existence of ν' can be expressed by a Σ_1-statement which is preserved by $\pi_{\sigma\tau}^{-1}$ and $\pi_{\sigma\lambda}$, contradicting the definition of ν.

For (c), let $s, p \in \operatorname{range} \pi_{\sigma\lambda}$ with $s \mathrel{\tilde{<}} s(\nu_\lambda)$ and $p \in L_s$. Let $\bar{s} = \pi^{-1}(s)$ and $\bar{p} = \pi^{-1}(p)$; observe that $\bar{s}, \bar{p} \in \operatorname{range} \pi_{\sigma\tau}$ and $\bar{p} \in L_{\bar{s}}$. Therefore, there exists a $\bar{\beta} \in \operatorname{range} \pi_{\sigma\tau}$ with $\bar{\beta} < \nu_\tau$ s. t. $L_{\bar{s}}\{\tau \cup \bar{p}\} \cap \nu_\tau \subset \bar{\beta}$. Then $\beta := \pi(\bar{\beta}) \in \operatorname{range} \pi_{\sigma\lambda}$ and $\beta < \nu_\lambda$. Finally, $L_s\{\lambda \cup p\} \cap \nu_\lambda \subset \beta$ as required.

For (d), we consider the cases that ν_τ is $\multimap$-minimal, successor, limit.

Before we start, note that we are already in a position to apply lemma 26 to $\pi_{\sigma\lambda}$. So if $\nu \multimap \nu_\tau$ (not necessarily immediate) with $\gamma_\nu \in \operatorname{range} \pi_{\sigma\tau}$ then $\bar{\nu} := \pi_{\sigma\tau}^{-1}(\nu) \multimap \nu_\sigma$ and $\pi_{\sigma\lambda}(\bar{\nu}) = \nu \multimap \nu_\lambda$.

Now assume that ν is the *immediate* $\multimap$-predecessor of ν_τ. We show that ν also is the *immediate* $\multimap$-predecessor of ν_λ.

Firstly assume $s(\nu_\lambda) \mathrel{\tilde{<}} s(\lambda)$, hence $s(\nu_\tau) \mathrel{\tilde{<}} \tilde{s}$, and let $\bar{\nu} = \pi_{\sigma\tau}^{-1}(\nu)$. Consider structures M_γ of the form $L^*_{s(\nu_\sigma)}\{\gamma \cup p_{\nu_\sigma}\}$ with $\gamma_{\bar{\nu}} < \gamma = L^*_{s(\nu_\sigma)}\{\gamma \cup p_{\nu_\sigma}\} \cap \sigma$. (Note that this implies $L^*_{s(\nu_\sigma)}\{\gamma \cup p_{\nu_\sigma}\} = L_{s(\nu_\sigma)}\{\gamma \cup p_{\nu_\sigma}\}$: Assume $x \in M_\gamma$, but then also $x \in L_{s(\nu_\sigma)} = L_{s(\nu_\sigma)}\{\sigma \cup p_{\nu_\sigma}\}$, hence there is a term t and $\vec{\xi} < \sigma$ s. t. $x = t(\vec{\xi}, p_{\nu_\sigma})$, therefore, $\vec{\xi} \in M_\gamma$ and hence $\vec{\xi} < \gamma$.) Let Γ be the set of all $\gamma < \sigma$ for which M_γ exists. If $\pi_\gamma \colon \bar{M}_\gamma \cong M_\gamma$ with $\nu^\gamma = \pi_\gamma^{-1}(\nu_\sigma)$ and $\bar{M}_\gamma = L_{s(\nu^\gamma)}$, then π_γ is a morass map (namely $\pi_{\nu^\gamma \nu_\sigma}$). But since $\bar{\nu}$ is the immediate $\multimap$-predecessor of ν_σ, this will not happen for any $\gamma \in \Gamma$. Furthermore, the fact that $\nu^{\gamma'}$ is not a $\multimap$-predecessor of ν_σ for all $\gamma' \in \Gamma \cap \gamma$ can be expressed in $\bar{M}_\gamma$. By type preservation for $\pi_{\sigma\lambda}$ and elementarity of $\pi_{\sigma\lambda} \restriction \bar{M}_\gamma$ we get that there are no $\multimap$-predecessors of ν_λ on levels between γ_ν and $\pi_{\sigma\lambda}(\gamma) =: \gamma^*$.

Consider the case that Γ is cofinal in σ. Since $\pi_{\sigma\lambda}$ is cofinal in λ, there are no $\multimap$-predecessors of ν_λ on levels greater than γ_ν. If $\Gamma \neq \emptyset$ is bounded below σ, let $\gamma = \max \Gamma$ (note that Γ is closed); otherwise let $\gamma = \gamma_{\bar{\nu}}$. Then

$L^*_{s(\nu_\sigma)}\{(\gamma+1)\cup p_{\nu_\sigma}\}\cap\sigma=\sigma$. Hence $L^*_{s(\nu_\lambda)}\{(\gamma^*+1)\cup p_{\nu_\lambda}\}\cap\lambda$ is cofinal in λ and there cannot be a $\mathrel{-\!\!3}$-predecessor of ν_λ on a level greater than γ^*. If $\gamma^*=\gamma_\nu$ we are done. Else we know that there cannot be a $\mathrel{-\!\!3}$-predecessor on a level between γ_ν and γ^*. Finally, γ^* cannot be the level of a $\mathrel{-\!\!3}$-predecessor of ν_λ because it is in range $\pi_{\sigma\lambda}$.

Secondly assume $s(\nu_\lambda)=s(\lambda)$ (hence $s(\nu_\tau)=s(\tau)$) and let α be the least ordinal s. t. $p_\tau\in L_{s(\tau)}\{(\alpha+1)\cup p_{\nu_\tau}\}$. Since p_τ is finite, $\alpha<\tau$ exists. Furthermore, $\gamma_\nu\leq\alpha$ for if $p_\tau\in L_{s(\tau)}\{\gamma_\nu\cup p_{\nu_\tau}\}=\operatorname{range}\pi_{\nu\nu_\tau}$ then $\operatorname{range}\pi_{\nu\nu_\tau}=L_{s(\tau)}$. Now it is clear that $L_{s(\tau)}\{(\alpha+1)\cup p_{\nu_\tau}\}=L_{s(\tau)}$. We want to show $\alpha\in\operatorname{range}\pi_{\sigma\tau}$. First find a term t s. t. there is a $\vec{\xi}$ s. t. $p_\tau=t(\vec{\xi},p_{\nu_\tau})$ and $\max(\vec{\xi})=\alpha$. Now the least $\vec{\xi}<\tau$ s. t. $p_\tau=t(\vec{\xi},p_{\nu_\tau})$ is in range $\pi_{\sigma\tau}$ and hence $\alpha\in\operatorname{range}\pi_{\sigma\tau}$, therefore $\alpha<\lambda$.

Let $\bar{\alpha}=\pi^{-1}_{\sigma\tau}(\alpha)$ and $L_{\bar{t}}\cong L_{s(\sigma)}\{\bar{\alpha}\cup p_{\nu_\sigma}\}$. By type preservation along $\pi_{\sigma\tau}$ we get $L_t\cong L_{s(\tau)}\{\alpha\cup p_{\nu_\tau}\}$ where $t:=\pi_{\sigma\tau}(\bar{t})$. By type preservation along $\pi_{\sigma\lambda}$ we get $L_t\cong L_{s(\lambda)}\{\alpha\cup p_{\nu_\lambda}\}$ and hence $L_{s(\tau)}\{\alpha\cup p_{\nu_\tau}\}\cong L_{s(\lambda)}\{\alpha\cup p_{\nu_\lambda}\}$.

Now the fact that there are no $\mathrel{-\!\!3}$-predecessors of ν_σ on levels (strictly) between $\pi^{-1}_{\sigma\tau}(\gamma_\nu)$ and $\bar{\alpha}$ is Σ_1 in $L_{s(\sigma)}$ (looking down to the situation in $L_{\bar{t}}$). Hence this is preserved to $L_{s(\lambda)}$. On levels (strictly) bigger than $\bar{\alpha}$ there cannot be a $\mathrel{-\!\!3}$-predecessor of ν_σ by definition of α. It remains to check if α can be a level of a $\mathrel{-\!\!3}$-predecessor of ν_λ if $\alpha>\gamma_\nu$. If it were, ν_σ would have had a $\mathrel{-\!\!3}$-predecessor on a level bigger than $\pi^{-1}_{\sigma\tau}(\gamma_\nu)=\gamma_{\bar{\nu}}$ by lemma 26. Therefore, ν is the largest $\mathrel{-\!\!3}$-predecessor of ν_λ.

Next consider the case that ν_τ is a $\mathrel{-\!\!3}$-limit. By iv)(e) of the morass map definition we have that $\{\gamma_\nu\mid\nu\mathrel{-\!\!3}\nu_\tau\}$ is cofinal in range $\pi_{\sigma\tau}\cap\tau=\lambda$. As we have seen above, this gives rise to $\mathrel{-\!\!3}$-predecessors of ν_λ, hence ν_λ is also a $\mathrel{-\!\!3}$-limit and, therefore, by iii) ((M3)) $\nu_\lambda\mathrel{-\!\!3}\nu_\tau$.

It remains the case that ν_τ is $\mathrel{-\!\!3}$-minimal. As above, if $s(\nu_\lambda)\mathrel{\tilde{<}}s(\lambda)$ and ν_λ had a predecessor, ν_τ also had one, contradicting our assumption. If $s(\nu_\lambda)=s(\lambda)$ look at the least α (as defined above) where we now have $\gamma_\tau<\alpha<\lambda$. As above, we cannot have levels of $\mathrel{-\!\!3}$-predecessors of ν_λ between γ_τ and α, at α, or between α and λ; hence ν_λ is also $\mathrel{-\!\!3}$-minimal.

We have seen that ν_τ is $\mathrel{-\!\!3}$-minimal, successor, limit iff ν_λ is $\mathrel{-\!\!3}$-minimal, successor, limit, respectively. For (d) it was therefore enough to show that ν_τ and ν_λ agree on their immediate $\mathrel{-\!\!3}$-predecessor ν.

It remains to show that $\sup\left(L_{s(\nu_\lambda)}\{\gamma_\nu \cup p_{\nu_\lambda}\} \cap \nu_\lambda\right) \in \operatorname{range}\pi_{\sigma\lambda}$. We know: $\nu'_\tau := \sup\left(L_{s(\nu_\tau)}\{\gamma_\nu \cup p_{\nu_\tau}\} \cap \nu_\tau\right) \in \operatorname{range}\pi_{\sigma\tau}$. But then $\pi(\nu'_\tau)$ is as required: If it were not a bound a counterexample would by structure-preservation along π^{-1} contradict that ν'_τ is a bound. If there were a smaller bound $\nu'_\lambda < \pi(\nu'_\tau)$ or $\pi^{-1}(\nu'_\lambda) < \nu'_\tau < \nu_\tau$, by iv)(f)ii we would have an $x \in H$ with $x \in L_{s(\nu_\tau)}\{\gamma_\nu \cup p_{\nu_\tau}\} \cap \nu'_\tau \setminus \pi^{-1}(\nu'_\lambda)$, contradicting that ν'_λ is a bound.

For (e), we have seen in the proof of (d) above, that ν_λ is a $\neg\exists$-limit iff ν_τ is a limit, in which case also ν_σ is a limit. Using corollary 27 we are done.

(f) is vacuous for $\pi_{\sigma\lambda}$. This concludes the proof of property iv). Hence $\pi_{\sigma\lambda}$ is a gap-2-morass map.

For $\sigma, \tau \in S^2$ this ends the proof. In the S^1 case: $\mu = \sup\pi_{\sigma\tau}[\nu_\sigma]$ clearly is a limit of ZF^--models. If $\mu = \nu_\tau$ then $L_\mu \models$ "$\tau = \aleph_i$ is the largest cardinal" else μ is a limit of such models. Similarly, $\bar\mu \in S^2$.

First we show $L_{s(\mu)} \cong L_{\hat s}\{\tau \cup p_{\nu_\tau}\}$, where $\hat s = \widetilde{\leq}$-lub $\left\{\pi_{\sigma\tau}(\bar t) \;\middle|\; \bar t \mathrel{\widetilde{<}} s(\nu_\sigma)\right\}$.

$L_{\hat s}\{\tau \cup p_{\nu_\tau}\} \cap \nu_\tau = \mu$: First assume $\xi_0 \in \mu$, then there is ξ_1 with $\xi_0 < \xi_1 < \mu$ and $\xi_1 = \pi_{\sigma\tau}(\bar\xi_1)$. Then $L_{\nu_\sigma} \models \overline{\overline{\bar\xi_1}} \leq \sigma$, hence exists $\bar f \in L_{\nu_\sigma}$ s. t. $\bar f \colon \sigma \twoheadrightarrow \bar\xi_1$. Then $\bar f \in L_{\bar t}\{\sigma \cup p_{\nu_\sigma}\}$ for some $\bar t \mathrel{\widetilde{<}} s(\nu_\sigma)$ with $\alpha(\bar t) < \alpha(s(\nu_\sigma))$. Let $f = \pi_{\sigma\tau}(\bar f) \in L_{\pi_{\sigma\tau}(\bar t)}\{\tau \cup p_{\nu_\tau}\}$, then $f \colon \tau \twoheadrightarrow \xi_1$, so $\xi_0 \in \operatorname{range} f$, hence $\xi_0 \in L_{\hat s}\{\tau \cup p_{\nu_\tau}\}$. On the other hand assume $\xi_0 \in L_{\hat s}\{\tau \cup p_{\nu_\tau}\} \cap \nu_\tau$, then there is a $\bar t \mathrel{\widetilde{<}} s(\nu_\sigma)$ s. t. $\xi_0 \in L_{\pi_{\sigma\tau}(\bar t)}\{\tau \cup p_{\nu_\tau}\}$. But $L_{\bar t}\{\sigma \cup p_{\nu_\sigma}\} \cap \nu_\sigma \subset \bar\beta$ for some $\bar\beta < \nu_\sigma$, hence also $L_{\pi_{\sigma\tau}(\bar t)}\{\tau \cup p_{\nu_\tau}\} \cap \nu_\tau \subset \pi_{\sigma\tau}(\bar\beta) < \mu$. So $\xi_0 \in \mu$ as required.

Let $\pi_0 \colon L_{\hat s}\{\tau \cup p_{\nu_\tau}\} \cong L_{s_0}$ and $p_0 = \pi_0(p_{\nu_\tau})$. Note that $\mu = \pi_0(\nu_\tau)$. To see that $s_0 = s(\mu)$, first note that s_0 singularizes μ, so $s(\mu) \mathrel{\widetilde{\leq}} s_0$. Assume for contradiction that s_0 is strictly greater. Then $p_\mu \in L_{s_0}\{\tau \cup p_0\}$, hence $p_\mu \in L_{s_1}\{\tau \cup p_0\}$ where $s(\mu) \mathrel{\widetilde{<}} s_1 \mathrel{\widetilde{<}} s_0$ (using that s_0 is a limit location). Since $L_{s(\mu)}\{\tau \cup p_\mu\} \subset L_{s_1}\{\tau \cup p_0\}$, s_1 singularizes μ. By definition of s_0, $\pi_0^{-1}(s_1) \mathrel{\widetilde{<}} \hat s$. Further, by definition of $\hat s$, there is a $\bar t \mathrel{\widetilde{<}} s(\nu_\sigma)$ s. t. $\pi_0^{-1}(s_1) \mathrel{\widetilde{\leq}} \pi_{\sigma\tau}(\bar t)$. By minimality of $s(\nu_\sigma)$, $L_{\bar t}\{\sigma \cup p_{\nu_\sigma}\} \cap \nu_\sigma \subset \bar\beta$ for some $\bar\beta < \nu_\sigma$. Hence $L_{\pi_{\sigma\tau}(\bar t)}\{\tau \cup p_{\nu_\tau}\} \cap \nu_\tau \subset \pi_{\sigma\tau}(\bar\beta) < \nu_\tau$. Since $\pi_0^{-1}(s_1) \mathrel{\widetilde{\leq}} \pi_{\sigma\tau}(\bar t)$, $L_{\pi_0^{-1}(s_1)}\{\tau \cup p_{\nu_\tau}\} \cap \nu_\tau \subset \pi_{\sigma\tau}(\bar\beta) < \nu_\tau$. Apply π_0: $L_{s_1}\{\tau \cup p_0\} \cap \mu \subset \pi_0 \circ \pi_{\sigma\tau}(\bar\beta) < \mu$, contradicting that s_1 singularizes μ.

Finally, observe that $p_0 = p_\mu$: $L_{s_\mu}\{\tau \cup p_0\}$ is cofinal in μ and hence $p_\mu \leq^* p_0$; assume for contradiction that p_μ is strictly smaller. Now by

$p_0 \in L_{s(\mu)} = L_{s(\mu)}\{\tau \cup p_\mu\}$ we have that $\pi_0^{-1}(p_\mu) <^* p_{\nu_\tau} \in L_{\hat{s}}\left\{\tau \cup \pi_0^{-1}(p_\mu)\right\} \subset L_{s(\nu_\tau)}\left\{\tau \cup \pi_0^{-1}(p_\mu)\right\}$, contradicting minimality of p_{ν_τ}.

Similarly, $\pi_1 \colon L_{\hat{\hat{s}}}\{\lambda \cup p_{\nu_\lambda}\} \cong L_{s(\bar{\mu})}$, where $\hat{\hat{s}} = \tilde{\leq}\text{-lub}\left\{\pi_{\sigma\lambda}(\bar{t}) \,\middle|\, \bar{t} \mathrel{\tilde{<}} s(\nu_\sigma)\right\}$, and $p_{\bar{\mu}} = \pi_1(p_{\nu_\lambda})$.

Let $\pi_2 := \pi_0 \circ \pi^{-1} \circ \pi_1^{-1} \colon L_{s(\bar{\mu})} \to L_{s(\mu)}$. We have $\pi_2 \restriction \lambda = \mathrm{id}$ (note that $\lambda = \gamma_{\bar{\mu}}$), $\pi_2(\lambda) = \tau = \gamma_\mu$, $\pi_2(\bar{\mu}) = \mu$, and $p_\mu = \pi_2(p_{\bar{\mu}}) \in \operatorname{range} \pi_2$.

Assume μ is a $\prec$-successor. Assume ν_τ is a $\prec$-limit, then so is ν_σ and hence also μ. So ν_τ must also be a $\prec$-successor with its predecessor in $\operatorname{range} \pi_{\sigma\tau}$. Therefore, $\mu = \nu_\tau$. But then their common predecessor is also in $\operatorname{range} \pi_2 \supset \operatorname{range} \pi_{\sigma\tau} \cap \nu_\tau$.

To see that π_2 is Σ_1-preserving, first consider $s \mathrel{\tilde{<}} s(\nu_\sigma)$ with $\alpha(s) = \alpha(\nu_\sigma)$ (as for $\tilde{s}$, $\hat{s}$ cannot be the first location on its level, hence this is true $s(\nu_\sigma)$). Then $L_s\{\sigma \cup p_{\nu_\sigma}\}$ is bounded in ν_σ, hence the collapse $L_s\{\sigma \cup p_{\nu_\sigma}\} \cong L_{\bar{s}}$ moves ν_σ. Using type preservation and elementary of a restriction of the respective morass maps we get as usual:

$$\pi_s^\lambda \colon L_{\pi_{\sigma\lambda}(s)}\{\lambda \cup p_{\nu_\lambda}\} \cong L_{\pi_{\sigma\lambda}(\bar{s})} = \pi_{\sigma\lambda}(L_{\bar{s}}) = \pi(\pi_{\sigma\tau}(L_{\bar{s}})) \text{ and}$$
$$\pi_s^\tau \colon L_{\pi_{\sigma\tau}(s)}\{\tau \cup p_{\nu_\tau}\} \cong L_{\pi_{\sigma\tau}(\bar{s})} = \pi_{\sigma\tau}(L_{\bar{s}}).$$

where π^{-1} is elementary between $L_{\pi_{\sigma\lambda}(\bar{s})}$ and $L_{\pi_{\sigma\tau}(\bar{s})}$. Note that we have

$$\bigcup_{s \tilde{<} s(\nu_\sigma)} L_{\pi_{\sigma\lambda}(s)}\{\lambda \cup p_{\nu_\lambda}\} = L_{\hat{\hat{s}}}\{\lambda \cup p_{\nu_\lambda}\} \cong L_{s(\bar{\mu})} \text{ and}$$
$$\bigcup_{s \tilde{<} s(\nu_\sigma)} L_{\pi_{\sigma\tau}(s)}\{\tau \cup p_{\nu_\tau}\} = L_{\hat{s}}\{\tau \cup p_{\nu_\tau}\} \cong L_{s(\mu)}.$$

Now, to see that π_2 is Σ_1-preserving, note that Σ_1-formulas are preserved upwards. For the other direction assume that φ is a QFF and $L_{s(\mu)} \models \exists v\, \varphi(v, \vec{r})$ where $\vec{r} \in \operatorname{range} \pi_2$. Let w be least s. t. $\varphi(w, \vec{r})$ and $s \mathrel{\tilde{<}} s(\nu_\sigma)$ s. t. $\pi_0^{-1}(w) \in L_{\pi_{\sigma\tau}(s)}\{\tau \cup p_{\nu_\tau}\}$. Then $\pi_s^\tau \circ \pi_0^{-1}(w) \in L_{\pi_{\sigma\tau}(\bar{s})}$ and hence $L_{\pi_{\sigma\tau}(\bar{s})} \models \exists v\, \varphi(v, \pi_s^\tau \circ \pi_0^{-1}(\vec{r}))$. This is preserved by π^{-1} to $L_{\pi_{\sigma\lambda}(\bar{s})}$. Finally expand using $\left(\pi_s^\lambda\right)^{-1}$ again and we get a witness in $\operatorname{dom} \pi_2$. Therefore, π_2 is the desired morass map $\pi_{\bar{\mu}\mu}$.

For $\pi_{\sigma\tau} \restriction \nu_\sigma = \pi_{\bar{\mu}\mu} \circ \pi_{\sigma\lambda} \restriction \nu_\sigma$, recall that $\pi_0 \restriction \mu = \mathrm{id} \restriction \mu$ and $\pi_1 \restriction \bar{\mu} = \mathrm{id} \restriction \bar{\mu}$. Hence $\pi_{\bar{\mu}\mu} \restriction \bar{\mu} = \pi^{-1} \restriction \bar{\mu}$ and $\pi_{\bar{\mu}\mu} \circ \pi_{\sigma\lambda} \restriction \nu_\sigma = \pi^{-1} \circ \pi_{\sigma\lambda} \restriction \nu_\sigma$. But we already know $\pi_{\sigma\tau} = \pi^{-1} \circ \pi_{\sigma\lambda}$.

v) is like (M7) in the gap-1 case. Only condition iv) for $\upsilon, \tau \in S^1$ is new in case $\sigma < \nu_\sigma$. We denote the maps from the gap-1 proof by $\pi_{\sigma\upsilon}$ and $\pi_{\upsilon\tau}$. (a), (b), (d), and (e) are clear since $\operatorname{range}\pi_{\sigma\tau} \subset \operatorname{range}\pi_{\upsilon\tau}$, and (f) is vacuous for $\pi_{\upsilon\tau}$.

For (c) let $s, p \in \operatorname{range}\pi_{\upsilon\tau}$ with $s \mathrel{\tilde{<}} s(\nu_\tau)$ and $p \in L_s$. We need to find a $\beta \in \operatorname{range}\pi_{\upsilon\tau}$ with $\beta < \nu_\tau$ and $L_s\{\tau \cup p\} \cap \nu_\tau \subset \beta$. First note that it is sufficient to show that for $s \in \operatorname{range}\pi_{\upsilon\tau}$ with $s \mathrel{\tilde{<}} s(\nu_\tau)$ and $p_{\nu_\tau} \in L_s$, there is a $\beta \in \operatorname{range}\pi_{\upsilon\tau}$ with $\beta < \nu_\tau$ and $L_s\{\tau \cup p_{\nu_\tau}\} \cap \nu_\tau \subset \beta$: For s, p there is an $s' \in \operatorname{range}\pi_{\upsilon\tau}$ with $s' \mathrel{\tilde{<}} s(\nu_\tau)$ s. t. $p \in L_{s'}\{\tau \cup p_{\nu_\tau}\}$ and hence $L_s\{\tau \cup p\} \subset L_{s'}\{\tau \cup p_{\nu_\tau}\}$.

Therefore, we now start with $s \in \operatorname{range}\pi_{\upsilon\tau}$ s. t. $s \mathrel{\tilde{<}} s(\nu_\tau)$. Let $\beta = \sup(L_s\{\tau \cup p_{\nu_\tau}\})$. Choose $t_0 \in \operatorname{range}\pi_{\upsilon\tau}$ s. t. $t_0 \mathrel{\tilde{<}} s(\tau)$ and $\beta, s, \tau, \nu_\tau, p_{\nu_\tau} \in L_{t_0}\{\gamma_\tau \cup p_\tau\} =: H$. Let $\bar{t}_0 = \pi_{\upsilon\tau}^{-1}(t_0)$, then $\bar{H} := L_{\bar{t}_0}\{\alpha \cup p_\upsilon\}$ is the corresponding hull an level α. Collapse and apply type preservation to $\bar{\varphi}\colon L_{\bar{t}_1} \cong \bar{H}$. We get $\varphi\colon L_{t_1} \cong H$ where $t_1 = \pi_{\upsilon\tau}(\bar{t}_1)$. This is a commutative diagram. We have $H' := L_{\varphi^{-1}(s)}\left\{\varphi^{-1}(\tau) \cup \varphi^{-1}(p_{\nu_\tau})\right\} \cap \varphi^{-1}(\nu_\tau) \subset \varphi^{-1}(\beta)$. Since $\pi_{\upsilon\tau} \restriction L_{\bar{t}_1}$ is elementary, $L_{\bar{\varphi}^{-1}(\bar{s})}\left\{\bar{\varphi}^{-1}(\upsilon) \cup \bar{\varphi}^{-1}(\bar{p})\right\} \cap \varphi^{-1}(\bar{\nu})$ is also bounded, where $\bar{s} = \pi_{\upsilon\tau}^{-1}(s)$, $\bar{p} = \pi_{\upsilon\tau}^{-1}(p_{\nu_\tau})$ and $\bar{\nu} = \pi_{\upsilon\tau}^{-1}(\nu_\tau)$. Let $\bar{\beta}'$ be the least upper bound. Then $\beta' := \pi_{\upsilon\tau}(\bar{\beta}')$ is the least upper bound on H' (again using elementarity). Finally, $\varphi(\beta') = \pi_{\upsilon\tau} \circ \bar{\varphi}(\bar{\beta}')$ is a bound on $L_s\{\tau \cup p_{\nu_\tau}\} \cap \nu_\tau$. This concludes the proof of property iv). Hence $\pi_{\upsilon\tau}$ is a gap-2-morass map.

In **vi)**, $\sigma < \nu_\sigma$ iff $\tau < \nu_\tau$ is corollary 24. The rest follows by full elementarity for a suitable restriction of $\pi_{\sigma\tau}$.

For **vii)** the first property follows from lemma 26. For the second property, if ν_τ is $\multimap 3$-minimal, then ν_σ is also $\multimap 3$-minimal by lemma 26. If ν_τ is a $\multimap 3$-successor, then ν_σ is also a $\multimap 3$-successor by lemma 28. If ν_τ is a $\multimap 3$-limit, then ν_σ is also a $\multimap 3$-limit by iv)(e) of the morass map definition and lemma 26. Still by lemma 26, $\bar{\nu} \multimap 3\ \nu_\sigma$ maps to $\nu \multimap 3\ \nu_\tau$. The diagram commutes by all the parameters matching and type preservation. The suprema match by lemma 29. The successor case was shown in lemma 28. $\dashv$

Remark 31 *Note that it follows from the proof of iv) that if* $\mu = \nu_\tau$, *hence* $\bar{\mu} = \nu_\lambda$, *then* μ *must be a* $-\!\!\!3$*-limit and* $\bar{\mu}$ *is as well: Since* $\bar{\mu} -\!\!\!3\, \mu$ *we cannot have that* μ, $\bar{\mu}$ *are* $-\!\!\!3$*-minimal or* $-\!\!\!3$*-successor (in the latter case they would have the same immediate predecessor). This says that if any morass map* $\pi_{\sigma\tau}$ *is cofinal into* ν_τ *and* ν_τ *is not a* $-\!\!\!3$*-limit, then* $\pi_{\sigma\tau}$ *is cofinal into* τ.

On the other hand (still for ν_τ *not a* $-\!\!\!3$*-limit), a morass map* $\pi_{\sigma\tau}$ *can be cofinal into* τ *but not into* ν_τ.

In Jensen's morass construction, for ν_τ *not a* $-\!\!\!3$*-limit, any morass map* $\pi_{\sigma\tau}$ *is cofinal into* τ *iff it is cofinal into* ν_τ. *This leads to a failure of perfect preservation. Jensen's morass obeys a dependency property which roughly says that if* ν_τ *is not a* $-\!\!\!3$*-limit then* τ *and* ν_τ *have the same collapsing structure. This is no longer true in hyperfine structure theory where* $s(\nu_\sigma)$ *may be strictly smaller than* $s(\sigma)$, *even when* ν_σ *is not a* $-\!\!\!3$*-limit.*

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