Diplom Thesis — Silver Machines

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May 5, 1997

Abstract

In this thesis I develop the concept of Silver machines. By means of a special machine — the Koepke-Richardson-Silver-machine — the Strong Covering Lemma is proved without fine structure theory.

Contents

0 Preliminaries

Introduction

Silver machines are a different view on the constructible universe introduced by Kurt Gödel in 1938. This concept of machines was first developed by Jack Silver, who used the word *machine* due to the motivation which led him to develop them. One might as well call it “Silver hierarchy”, since Silver machines describe when sets are created — similar to the Jensen hierarchy. An approach based on similar ideas is due to Friedman and Koepke [4]. This thesis relies very much on the PhD thesis of Thomas Lloyd Richardson [5], a student of Silver, as well as several talks of my advisor, Peter Koepke.

The constructible universe L of set theory is defined as the class of sets definable in a transfinite process as follows: We start with an empty L_0, for L_α already defined let $L_{\alpha+1}$ consist of all subsets of L_α definable by $\in$-formulae, and for limit ordinals λ we take the union of all previous stages of the construction, $L_\lambda = \bigcup_{\alpha<\lambda} L_\alpha$. Finally $L = \bigcup_{\alpha\in\mathrm{On}} L_\alpha$.
L turns out to be the smallest model of set theory, i.e., it is a subclass of every other model. As a consequence of the very concrete definition, L has some fundamental properties undecidable in ZFC alone. Actually, Gödel defined this model to prove the consistency of ZFC and the continuum hypothesis (CH). This proof is based on the *condensation lemma* which states that Σ_1-substructures of L condense down to stages of L.

In contrast to this simplicity, there are more complicated statements, e.g., combinatorial principles (as the $\Box$-principle) or the Covering Lemma. It was Ronald Jensen who in 1972 came up with a solution for this kind of problems, the so-called *fine structure theory*. The idea is to have a closer look at the passage from L_α to $L_{\alpha+1}$ and to describe the process with some “bookkeeping device”. Even today — after 25 years of development —, this

method is extremely complicated and wearisome.

Also in the early seventies, Jack Silver found a different approach — the Silver machines. These machines reduce the considerations to calculations with sets of ordinals. Similar as for L, a new hierarchy for these sets, M^δ, is defined. Analogically to the condensation lemma we have a *collapsing property*, i.e., closed structures (which are produced by a hull operator) condense down to stages of the machine. In contrast to the constructible universe very little happens in the passage from M^δ to $M^{\delta+1}$. This is guaranteed by a certain *finiteness property* which codes all information needed for this step in a finite set which itself has a simple form. This basic theory is developed in the first chapter.

The Koepke-Richardson-Silver-machine defined in the second chapter will consist of functions used for coding and decoding sequences of ordinals, a truth function, and a Skolem function. The later relate the machine to the situation in L.

The Covering Lemma* states that — under the assumption that $0^\sharp$ does not exist — every uncountable set of ordinals is covered by a set of the same cardinality in L. This is proved in the third chapter as an application of the Koepke-Richardson-Silver-machine. Finally, we prove the Strong Covering Lemma, a strengthening of the above, namely an extension to structures.

Notations

The basic concepts of set theory (especially the constructible universe L) are assumed to be known. Any notation and definition not explained is standard and may, e.g., be found in Drake [2].

We use the usual logical symbols: $\wedge$ (and), $\vee$ (or), $\neg$ (not), $\exists$ (exists), $\forall$ (for

*Traditionally named lemma, theorem would be more adequate and is used occasionally. However, we side with the tradition.

all), $\rightarrow$ (implies), (, and) (parentheses)

For two sets x and y we write $x \cong y$ if x and y are isomorphic (i.e., there exists a 1-1 function from x onto y which preserves all structures on x; the structures will be clear from the context). Furthermore, we write $x \subset y$ if x is a (not necessarily proper) subset of y. For the set of all subsets of x with cardinality κ we write $[x]^\kappa$. For a set of ordinals X let $\operatorname{lub} X$ (least upper bound) be the least δ with $X \subset \delta$. As usual small greek letters will denote ordinals.

Let $f\colon x \rightarrow y$; we write $\operatorname{dom} f$ for the domain and $\operatorname{range} f$ for the range of f s.t. $f \subset \operatorname{range} f \times \operatorname{dom} f$. The set of all functions f with domain x and $\operatorname{range} f \subset y$ will be denoted by ${}^x y$. ${}^{<\omega}x$ is the set of all finite sequences in x. If x and y are ordered sets and f is a bijection which preserves this we write $f\colon x \xleftrightarrow{\text{o.p.}} y$. Finally, f is a partial function $f\colon x \rightharpoonup y$ if $\operatorname{dom} f \subset x$. For two partial functions f and g we write $f(x) \simeq g(x)$ to mean that $f(x)$ is defined iff $g(x)$ is defined, in which case they are equal. Furthermore, $f(x){\downarrow}$ iff f is defined at x, else $f(x){\uparrow}$.

$\langle B, \leq\rangle$ is called directed partial ordering if it is a partial ordering (a transitive and reflexive relation) and $\forall a, b \in B \ \exists c \in B\, (a \leq c \wedge b \leq c)$. Let $Z = \langle \gamma_b, \in\rangle_{b\in B}, \langle \pi_{b_1 b_2}\rangle_{b_1 \leq b_2}$ be a commutative system of maps, i.e., $\pi_{b_1 b_2}\colon \langle \gamma_{b_1}, \in\rangle \rightarrow \langle \gamma_{b_2}, \in\rangle$ is order-preserving and if $b_1 \leq b_2 \leq b_3$ then $\pi_{b_1 b_3} = \pi_{b_2 b_3} \circ \pi_{b_1 b_2}$. Then $\langle A, E, \pi_b\rangle_{b\in B}$ is a direct limit of the directed system Z iff

i) $\pi_b\colon \langle \gamma_b, \in\rangle \rightarrow \langle A, E\rangle$ is order-preserving,

ii) $A = \bigcup_{b\in B} \operatorname{range} \pi_b$ and

iii) if $b_1 \leq b_2$ then $\pi_{b_2} \circ \pi_{b_1 b_2} = \pi_{b_1}$.

We call this system well-founded iff $\langle A, E\rangle$ is well-founded. It is a well-known fact that any two direct limits of a system are isomorphic; so, in case $\langle A, E\rangle$ is well-founded, we may take $\langle \gamma, \in\rangle$ as *the* direct limit where γ is the transitive collapse of $\langle A, E\rangle$.

1 Silver Machines

We commence with a series of definitions. First the general notion of a machine and the fundamental closure- or hull-operation.

Definition 1 (Machines) A sequence $M = (\mathrm{On}, <, M_i)_{i<\omega}$ where M_i is a (class) partial function $M_i \colon {}^{<\omega}\mathrm{On} \rightharpoonup \mathrm{On}$ (all $i < \omega$) is called *machine*.

For $\delta \in \mathrm{On}$ we define $M^\delta = \left(\delta, < \cap\, \delta^2, M_i^\delta\right)_{i<\omega}$ where $M_i^\delta = M_i \cap (\delta \times {}^{<\omega}\delta)$.

By this definition, a machine is essentially a countable sequence of functions. Intuitively, these functions may be seen as algorithms for creating new sets from an input. The "stages" M^δ of this machine restrict the calculations, such that we have sets instead of classes. The idea of the stages is, that, when the stage M^δ of the machine is applied (as an operator) to a set X of ordinals, we get all sets which are calculated from X in δ-many steps.

Definition 2 (Closure) For $\delta \in \mathrm{On}$ and $X \subset \delta$ let

$$M^\delta[X] = \bigcap \left\{ Y \,\middle|\, X \subset Y \subset \delta \wedge \forall i \in \omega \; M_i^\delta \left[{}^{<\omega}Y\right] \subset Y \right\}.$$

$M^\delta[X]$ is the *closure* of the set X under the stage δ of the machine M.

For a machine M and an ordinal δ, a set X is called M^δ*-closed* ($\mathrm{cl}_M^\delta(X)$) iff $X \subset \delta$ and $M^\delta[X] = X$.

Now we can explain what we mean by isomorphic in respect to machines: For some δ and an M^δ-closed set X we write $M^{\bar\delta} \cong X$ iff there exists an order-preserving 1-1 function π from $M^{\bar\delta}$ onto X with

$$\forall i < \omega \; \forall u \in {}^{<\omega}\bar\delta \;\; \pi\Big(M_i^{\bar\delta}(u)\Big) \simeq M_i^\delta\left(\pi(u)\right).$$

The next property describes a structure of a machine which grows slowly and coherently. It will define the notion of Silver machines.

Definition 3 (Finite Support Property — FSP) A machine M satisfies the *finite support property* (*FSP*) iff for all $\delta \in \mathrm{On}$ there is a finite set $H_\delta \subset \delta$ s.t.:

i) $H_\delta \subset M^{\delta+1}[\{\delta\}]$

ii) If X is M^δ-closed with $H_\delta \subset X$ and $M^{\bar{\delta}} \cong X$, then $X \cup \{\delta\}$ is $M^{\delta+1}$-closed with $M^{\bar{\delta}+1} \cong (X \cup \{\delta\})$.

Intuitively, this means that the step from stage δ to stage $\delta+1$ can be coded by only finitely many ordinals and that the calculations from earlier stages are preserved.

Definition 4 (Silver Machine) A machine which satisfies FSP is called *Silver machine*.

In the rest of this chapter we work out some consequences of this definition, you could say we split FSP in handy properties. The first is the analog to the condensation lemma in L. Roughly speaking, the collapsing property says that closed structures are of a simple, coherent form.

Definition 5 (Collapsing Property) A machine M satisfies the *collapsing property* iff, whenever X is M^δ-closed (some $\delta \in \mathrm{On}$), then $M^{\bar{\delta}} \cong X$ where $\bar{\delta} = \mathrm{otp}(X)$.

Theorem 1 *Silver machines satisfy the collapsing property.*

Proof Given $\delta \in \mathrm{On}$ and an M^δ-closed X let $\bar{\delta} = \mathrm{otp}(X)$ and $\pi \colon \bar{\delta} \xleftrightarrow{\text{o.p.}} X$. Extend π by $\pi\left(\bar{\delta}\right) = \delta$. First note the following trivial fact:

$$\forall \bar{\eta} \leq \bar{\delta}\ \mathrm{cl}_M^{\pi(\bar{\eta})}\left(X \cap \pi\left(\bar{\eta}\right)\right)$$

To see this note that $M^{\pi(\bar{\eta})}\left[X \cap \pi\left(\bar{\eta}\right)\right] \subset M^\delta\left[X \cap \pi\left(\bar{\eta}\right)\right] \cap \pi\left(\bar{\eta}\right) \subset X \cap \pi\left(\bar{\eta}\right)$.

We show by induction: $\forall \bar{\eta} \leq \bar{\delta}\ M^{\bar{\eta}} \cong X \cap \pi\left(\bar{\eta}\right)$ (This implies $M^{\bar{\delta}} \cong X$.) For $\bar{\eta} = 0$ and $\lim \bar{\eta}$ this is clear. So assume $\bar{\eta} = \bar{\mu} + 1$ and let $\mu = \pi\left(\bar{\mu}\right) \in X$.

By the induction hypothesis we have $M^{\bar{\mu}} \cong X \cap \mu$. By the fact we know $\mathrm{cl}^{\mu}_{M}(X \cap \mu)$. By FSP there is an $H_\mu \subset M^{\mu+1}[\{\mu\}] \cap \mu \subset X \cap \mu$. Hence $M^{\bar{\mu}+1} \cong (X \cap \mu) \cup \{\mu\} = X \cap \pi(\bar{\mu}+1)$ and we are done. $\dashv$

Remark 1 *Let M be a machine which satisfies the collapsing property. If $\delta \in \mathrm{On}$ and $X \subset \delta$ then $\pi\colon M^{\bar{\delta}} \cong M^{\delta}[X]$ and $\bar{\delta} = M^{\bar{\delta}}\left[\bar{X}\right]$, where $\bar{\delta} = \mathrm{otp}\left(M^{\delta}[X]\right)$, $\pi\colon \bar{\delta} \xleftrightarrow{o.p.} M^{\delta}[X]$, and $\pi\left[\bar{X}\right] = X$.*

Proof Assume a situation as in the remark. By the theorem above we infer $M^{\bar{\delta}} \cong M^{\delta}[X]$. Let $\bar{Y} = M^{\bar{\delta}}\left[\bar{X}\right]$ and $Y = \pi\left[\bar{Y}\right]$. Then Y is M^{δ}-closed with $X \subset Y$, so $M^{\delta}[X] \subset Y$. Hence $\bar{Y} = \bar{\delta}$. $\dashv$

Definition 6 (Collapsing Function, Structure-Preserving) Let X be M^{δ}-closed for some $\delta \in \mathrm{On}$ and $\pi\colon M^{\bar{\delta}} \cong X$. Then we call π *collapsing function*. In this case we also write $\pi\colon \bar{\delta} \cong X$ ($=$ range π) or $\pi\colon \bar{\delta} \cong \delta$ (range π is M^{δ}-closed). If $\pi\colon \bar{\delta} \to \delta$ and range π is $M^{\delta'}$-closed for some $\delta' \leq \delta$ then π is called *structure-preserving*.

Note that the following are equivalent by the collapsing property:

i) $\pi\colon \bar{\delta} \cong \delta$

ii) π is order-preserving and $\forall i < \omega\ \forall u \in {}^{<\omega}\bar{\delta}\ \pi\Big(M_i^{\bar{\delta}}(u)\Big) \simeq M_i^{\delta}(\pi(u))$.

iii) π is order-preserving and $M^{\delta}[\mathrm{range}\,\pi] \subset \mathrm{range}\,\pi$.

Furthermore, $\pi\colon \bar{\delta} \to \delta$ is structure-preserving iff π is order-preserving and for $i < \omega$ and $u \in {}^{<\omega}\bar{\delta}$ if $M_i^{\bar{\delta}}(u)\downarrow$ then $M_i^{\delta}(\pi(u))\downarrow$ and $\pi\Big(M_i^{\bar{\delta}}(u)\Big) = M_i^{\delta}(\pi(u))$.

Proposition 1

i) Each collapsing function is structure-preserving.

ii) Let $\pi_0\colon \delta_0 \cong \delta_1$ and $\pi_1\colon \delta_1 \cong \delta_2$. Then $\pi_1 \circ \pi_0\colon \delta_0 \cong \delta_2$.

iii) Let $\pi_0 \colon \delta_0 \cong \delta_1$ *and* $\pi_1 \colon \delta_2 \cong \delta_3$ *where* $\operatorname{range} \pi_0 \subset \operatorname{range} \pi_1$. *Then* $\pi_1^{-1} \circ \pi_0 \colon \delta_0 \cong \delta_2$.

iv) If $\pi \colon \delta_0 \to \delta_1$ *is structure-preserving then* $\pi \colon \delta_0 \cong \operatorname{lub}(\operatorname{range} \pi)$.

v) Let $\pi_0 \colon \delta_0 \to \delta_1$ *and* $\pi_1 \colon \delta_2 \to \delta_3$ *be structure-preserving where* $\operatorname{range} \pi_0 \subset \operatorname{range} \pi_1$. *Then* $\pi_1^{-1} \circ \pi_0 \colon \delta_0 \to \delta_2$ *is structure-preserving.* $\dashv$

i) – iii) are clear by definition. For iv) note that by definition $\mathrm{cl}_M^{\delta_2}(\operatorname{range} \pi)$ for some $\delta_2 \leq \delta_1$. Clearly, $\operatorname{lub}(\operatorname{range} \pi) \leq \delta_2$. Hence $M^{\operatorname{lub}(\operatorname{range} \pi)}[\operatorname{range} \pi] \subset M^{\delta_2}[\operatorname{range} \pi] \subset \operatorname{range} \pi$ and, therefore, $\mathrm{cl}_M^{\operatorname{lub}(\operatorname{range} \pi)}(\operatorname{range} \pi)$. v) is immediate from iii) and iv).

The next property says that only the knowledge of a finite set of ordinals is necessary to understand what happens at one additional step of calculations.

Definition 7 (Finiteness Property) A machine M satisfies the *finiteness property* iff for all $\delta \in \mathrm{On}$ there is a finite set $F_\delta \subset \delta$ s.t. for all $X \subset \delta+1$, $M^{\delta+1}[X] \subset M^{\delta}[(X \cap \delta) \cup F_\delta] \cup \{\delta\}$.

Theorem 2 *Silver machines satisfy the finiteness property.*

Proof Choose $F_\delta = H_\delta$. Then the right hand side is $M^{\delta+1}$-closed by FSP. Since X is a subset of the right hand side, the same is true for the $M^{\delta+1}$-closure. $\dashv$

Definition 8 (Direct Limit Property) A machine M satisfies the *direct limit property* iff the following holds: Let $Z = \langle \gamma_b, \in \rangle_{b \in B}, \langle \pi_{b_1 b_2} \rangle_{b_1 \leq b_2}$ be a well-founded directed system with direct limit $\langle \gamma, \in, \pi_b \rangle_{b \in B}$. If each $\pi_{b_1 b_2}$ is structure-preserving so is π_b.

Theorem 3 *Silver machines satisfy the direct limit property.*

Proof First note that each π_b is order-preserving by definition of the direct limit. Let $X_b = \operatorname{range}(\pi_b)$ and extend π_b by $\pi_b(\gamma_b) = \operatorname{lub} X_b$. We show $M^{\operatorname{lub} X_b}[X_b] \subset X_b$.

For all $b \in B$ we proceed simultaneously by induction on $\alpha \leq \gamma$ to show that, if for some $\alpha_b \leq \gamma_b$ we have $\pi_b(\alpha_b) = \alpha$, then $\pi_b \restriction \alpha_b \colon \alpha_b \cong \alpha$, i.e., $M^\alpha[X_b \cap \alpha] \subset X_b \cap \alpha$.

For $\alpha = 0$ this is clear. For any $i < \omega$ and $u \in {}^{<\omega}\alpha_b$ s.t. $\beta := M_i^\alpha(\pi_b(u))\downarrow$ we need to show that $\beta \in X_b \cap \alpha$. Let $\mu = \max(\pi_b(u)) \cup \beta < \alpha$ and choose $b' \geq b$ s.t. $\beta = \pi_{b'}(\beta')$ for some $\beta' < \gamma_{b'}$ and $H_\mu \subset X_{b'}$. (This can be done since for each $\delta \in \gamma$ there is a $c \in B$ s.t. $\delta \in \text{range}(\pi_c)$ and, therefore, $\delta \in \text{range}(\pi_{c'})$ for any $c' \geq c$, and since H_μ is finite.) Now let $u' = \pi_{bb'}(u)$ (so $\pi_{b'}(u') = \pi_b(u)$) and $\mu' = \pi_{b'}^{-1}(\mu) \leq \mu < \alpha$. By the induction hypothesis and by FSP $\pi_{b'} \restriction (\mu'+1) : (\mu'+1) \cong (\mu+1)$. It follows that $M_i^{\mu'+1}(u') = \beta'$.

Since $\beta < \alpha \leq \text{lub}\, X_b$ we have $\beta \leq \xi$ for a $\xi \in X_b$ (else $\beta \geq \text{lub}\, X_b$). Let $\xi_b < \gamma_b$ s.t. $\pi_b(\xi_b) = \xi$ and $\xi_{b'} = \pi_{bb'}(\xi_b)$. Therefore, $\pi_{b'}(\xi_{b'}) = \xi$ and, finally, $\beta' \leq \xi_{b'} \in \text{range}\, \pi_{bb'}$. So, $\mu' < \text{lub}(\text{range}\, \pi_{bb'})$. Now $\pi_{bb'} \colon \gamma_b \cong \text{lub}(\text{range}\, \pi_{bb'})$ and we can conclude that $\pi_{bb'}(\bar\beta) = \beta'$ for some $\bar\beta < \alpha_b$. Then $\beta = \pi_{b'} \circ \pi_{bb'}(\bar\beta) = \pi_b(\bar\beta) \in X_b \cap \alpha$ as desired. $\dashv$

2 The KRS-Machine

In this chapter we construct a Silver machine suitable for L and the proofs of the Covering Lemmas.

First we need to define our language.

Definition 9 (Language, Terms, Formulas) Let $\mathcal{L}$ consist of the following *symbols*: $\in, =, \vee, \neg, (,)$; variables v_i for all $i \in \omega$; quantifiers $\exists_\alpha and S_\alpha$ for all $\alpha \in \mathrm{On}$.

The following inductively defines *terms* and *formulas*:

i) If s and t are variables or terms, then $(s = t)$ and $(s \in t)$ are formulas. We do not allow the case that s or t is a variable which occurs as a bounded variable in t respectively s.

ii) If φ and ψ are formulas, then $(\varphi \vee \psi)$, $(\neg\, \varphi)$, and $(\exists_\alpha v_i)\, \varphi$ are formulas. For the $\vee$-case, we do not allow that a variable free in φ is bounded in ψ or vice versa. For the $\exists_\alpha$-case, v_i must not be bounded in φ.

iii) If φ is a formula, v_i is the only free variable in φ, and neither $\exists_\beta$ for a $\beta > \alpha$ nor S_β for a $\beta \geq \alpha$ is a symbol in φ (i.e., φ has rank $\leq \alpha$, see below), then $(S_\alpha v_i)\, \varphi\, (v_i)$ is a term.

By definition, terms have no free variables. Furthermore, a variable occurring bounded within a formula cannot be free anywhere else in the same formula; this is done so that we can easily substitute free variables. Of course, this restriction is only technical. The logical symbols not specified above (such as $\forall_\alpha$, $\leftrightarrow$, ...) will be defined in the obvious way.

Definition 10 (Rank) For a term or a formula φ let the *rank* of φ, $\mathrm{rk}(\varphi)$, be the least α s.t.

i) if $\exists_\beta$ occurs in φ, then $\beta \leq \alpha$ and

ii) if S_β occurs in φ, then $\beta < \alpha$.

The idea behind these definitions is that L_α consists of the interpretations of the terms of rank $\leq \alpha$ and that $(\exists_\alpha v_i)\,\varphi$ means $\exists v_i \in L_\alpha\ \varphi$. A suitable assignment will be introduced later in this chapter. As a next step, we need some tools to code ("Gödel-number") our language.

Definition 11 (Pairing Function) A *pairing function* is a 1-1 onto function $P\colon {}^{<\omega}\mathrm{On} \to \mathrm{On}$ s.t. $\forall u \in {}^{<\omega}\mathrm{On}\ P(u) \geq \max(u)$. For $u, v \in {}^{<\omega}\mathrm{On}$ let $u <_P v$ iff $P(u) < P(v)$. Define "inverse functions" $P_k\colon \mathrm{On} \rightharpoonup \mathrm{On}$, $k \in \omega$, by

$$P_k(\delta) = \begin{cases} \alpha_k, & \text{if } P^{-1}(\delta) = \langle \alpha_0, \ldots, \alpha_{n-1} \rangle \text{ and } k < n \\ \uparrow\ , & \text{else} \end{cases}.$$

From now on we fix the following pairing function P:
For $u \in {}^{<\omega}\mathrm{On}$ and $\alpha \in \mathrm{On}$ let $\mathrm{occ}\,(\alpha, u) =$ the number of occurrences of α in u. Define $u <_P v$ iff

i) $\mathrm{occ}\,(\alpha, u) < \mathrm{occ}\,(\alpha, v)$, where α is the largest ordinal s.t. $\mathrm{occ}\,(\alpha, u) \neq \mathrm{occ}\,(\alpha, v)$ or

ii) $\forall \alpha\ \ \mathrm{occ}\,(\alpha, u) = \mathrm{occ}\,(\alpha, v)$ and $u <_{lex} v$.

Obviously, $<_P$ is a (strict) linear ordering. Note that $<_P$ is also well-founded: Assume $\langle u_i \rangle_{i<\omega}$ is a $<_P$-decreasing sequences of elements of ${}^{<\omega}\mathrm{On}$. Since for each u_i there are only finitely many $u \in {}^{<\omega}\mathrm{On}$ s.t. $\forall \alpha\ \ \mathrm{occ}\,(\alpha, u_i) = \mathrm{occ}\,(\alpha, u)$, we may assume that all u_i are ordered by condition i). Then let $\alpha_i = \max\,(u_i)$ for all $i < \omega$; $\langle \alpha_i \rangle$ is (not necessarily strictly) decreasing and hence has a minimum, α say. Now take the subsequence of $\langle u_i \rangle_{i<\omega}$ with $\max\,(u_i) = \alpha$; of those take all with minimal $\mathrm{occ}\,(\alpha, u_i)$. All elements of this subsequence must be smaller than the rest by construction. Ignoring α, we can repeat finding the minimal maximum and so on. Since these maxima are a strictly decreasing sequence of ordinals, this process stops after finitely many steps and we are done. So $P(u) = \mathrm{otp}_{<_P}(u)$ defines a 1-1 onto function $P\colon {}^{<\omega}\mathrm{On} \to \mathrm{On}$.

It remains to show that $\forall u \in {}^{<\omega}\mathrm{On}\ P(u) \geq \max(u)$. Assume not; then choose the $<_P$-minimal u with $\alpha := P(u) < \beta := \max(u)$. Hence $u = \langle\beta\rangle$. But $P(u) > P(\langle\alpha\rangle) \geq \alpha$ which is a contradiction. Therefore, P is a pairing function as required.

Definition 12 (*P*-closed) An ordinal δ is P*-closed* iff $\forall u \in {}^{<\omega}\delta\ P(u) < \delta$.

Definition 13 (Coding) The symbols of $\mathcal{L}$ are identified with ordinals as follows:

symbol	$\in$	$=$	$\vee$	$\neg$	(	)	v_i	$\exists_\alpha$	S_α
ordinal	0	1	2	3	4	5	$i+6$	$P(\langle 1, \alpha\rangle)$	$P(\langle 0, 1, \alpha\rangle)$

So any term or formula φ may be seen as a sequence s of ordinals. Let $\ulcorner\varphi\urcorner = P(s)$. In this context we identify each ordinal σ with the sequence $\langle\sigma\rangle$ so that $\ulcorner\sigma\urcorner = \ulcorner\langle\sigma\rangle\urcorner$.

The following is immediate from the definition:

Proposition 2

i) For all $\alpha, \beta \in \mathrm{On}$ with $\alpha < \beta$: $\omega < \exists_\alpha < S_\alpha < \exists_\beta$

ii) If φ is a formula or term and σ a symbol occurring in φ, then $\ulcorner\sigma\urcorner < \ulcorner\varphi\urcorner$.

iii) If φ is a formula or term of rank α and β is the least P-closed ordinal greater than α, then $\alpha < \ulcorner\varphi\urcorner < \beta$.

iv) If φ and ψ are formulas where φ is a proper subformula of ψ, then $\ulcorner\varphi\urcorner < \ulcorner\psi\urcorner$.

v) If t is a term and φ a formula containing t, then $\ulcorner t\urcorner < \ulcorner\varphi\urcorner$.

vi) If s and t are terms with $\mathrm{rk}(s) < \mathrm{rk}(t)$, then $\ulcorner s\urcorner < \ulcorner t\urcorner$.

vii) If t is a term of rank $\leq \alpha$ and $\varphi(v_i)$ is a formula, then $\ulcorner t\urcorner < \ulcorner(\exists_\alpha v_i)\,\varphi(v_i)\urcorner$ and $\ulcorner\varphi[t]\urcorner < \ulcorner(\exists_\alpha v_i)\,\varphi(v_i)\urcorner$.

viii) If, for some $\delta \in \mathrm{On}$, X is M^δ-closed, where the pairing function P is a function of M, and $\pi\colon M^{\bar\delta} \cong X$ then: there exists a term/formula φ with $\ulcorner\varphi\urcorner \in \bar\delta$ iff there exists a term/formula ψ with $\ulcorner\psi\urcorner \in X$. (In fact ψ is obtained by replacing every occurrence of $\exists_\alpha$ with $\exists_{\pi(\alpha)}$ and S_α with $S_{\pi(\alpha)}$.) $\dashv$

E.g., note for viii) that $\pi\restriction\left(\bar\delta \cap \omega\right) = \mathrm{id}$. This is clear since X is M^δ-closed and P applied to the sequence of n 0s equals n, therefore, $\delta \cap \omega \subset X$.

The next definition will simulate the situation in L. At the end of this chapter the machine is mapped onto L.

Definition 14 (Truth Function and Skolem Function) We define a *truth function* $T\colon \{\ulcorner\varphi\urcorner \,|\, \varphi \text{ sentence}\} \to 2$ by induction on rank. Assume this is done for all $\beta < \alpha$ and φ is a sentence of rank α. Then $T(\varphi)$ is defined by induction on the complexity of φ as follows:

i) For terms $s = (S_\beta v_i)\,\psi\,(v_i)$ and $t = (S_\gamma v_j)\,\chi\,(\psi_j)$ (let k, m, and n — all different — be minimal s.t. v_k, v_m, and v_n do not occur in ψ and χ):

$$T\left(\ulcorner s = t\urcorner\right) = \begin{cases} T\left(\ulcorner(\forall_\beta v_k)\left(\psi\left(v_k\right) \leftrightarrow \chi\left(v_k\right)\right)\urcorner\right) & ,\beta = \gamma \\ T\left(\ulcorner(\forall_\gamma v_k)\left(\psi\left(v_k\right) \leftrightarrow \chi\left(v_k\right) \wedge v_k \in l_\beta\right)\urcorner\right), & \beta < \gamma \\ T\left(\ulcorner(\forall_\beta v_k)\left(\psi\left(v_k\right) \wedge v_k \in l_\gamma \leftrightarrow \chi\left(v_k\right)\right)\urcorner\right), & \beta > \gamma \end{cases} \text{ and}$$

$$T\left(\ulcorner s \in t\urcorner\right) =$$
$$= \begin{cases} T\left(\ulcorner(\exists_\gamma v_k)\left(\chi\left(v_k\right) \wedge (\forall_\beta v_m)\left(v_m \in v_k \leftrightarrow \psi\left(v_m\right)\right)\right)\urcorner\right) & ,\beta \geq \gamma \\ T\left(\ulcorner(\exists_\gamma v_k)\left(\chi\left(v_k\right) \wedge (\forall_\gamma v_m)\left(v_m \in v_k \leftrightarrow \psi\left(v_m\right) \wedge v_m \in l_\beta\right)\right)\urcorner\right), & \beta < \gamma \end{cases},$$

where l_β is the term $(S_\beta v_n)\left(v_n = v_n\right)$.

ii) Connectives:

$T\left(\ulcorner\neg\,\psi\urcorner\right) = 1 \setminus T\left(\ulcorner\psi\urcorner\right)$

$T\left(\ulcorner\psi_1 \vee \psi_2\urcorner\right) = T\left(\ulcorner\psi_1\urcorner\right) \cup T\left(\ulcorner\psi_2\urcorner\right)$

iii) Quantifier ($\beta \leq \alpha$):

$T\left(\ulcorner(\exists_\beta v_i)\,\psi\left(v_i\right)\urcorner\right) = \bigcup\left\{T\left(\ulcorner\psi[t]\urcorner\right) \,|\, t \text{ term} \wedge \mathrm{rk}(t) \leq \beta\right\}$

We say that φ is true iff $T\left(\ulcorner\varphi\urcorner\right) = 1$.

Finally, we define a *Skolem function*

$$h\colon \left\{\ulcorner\varphi\urcorner \middle| T\left(\ulcorner\varphi\urcorner\right) = 1 \wedge \varphi = \left(\exists_\alpha v_i\right)\psi\left(v_i\right) \text{ some } \alpha, i, \psi\right\} \to \mathrm{On}$$

s.t. if $\varphi \in \operatorname{dom} h$ then $\psi[t]$ is true where t is minimal with this property and $\ulcorner t\urcorner = h\left(\ulcorner\varphi\urcorner\right)$.

Now we are ready to define the desired machine and prove that it is actually a Silver machine.

Definition 15 (KRS-Machine) We define the *KRS-machine* (*Koepke-Richardson-Silver-machine*) as $M = \left(\mathrm{On}, <, P, P_k, T, h\right)_{k<\omega}$.

Theorem 4 *The KRS-machine is a Silver machine.*

Proof We show that $H_\delta = \left(\left\{P_k(\delta) \middle| k < \omega\right\} \cup \{h(\delta)\}\right) \cap \delta$ is as required. Obviously H_δ is finite and $H_\delta \subset M^{\delta+1}\left[\{\delta\}\right]$. So suppose X is M^δ-closed with $H_\delta \subset X$ and $M^{\bar\delta} \cong X$. We need to show that $X \cup \{\delta\}$ is $M^{\delta+1}$-closed with $M^{\bar\delta+1} \cong \left(X \cup \{\delta\}\right)$. That is for $\pi\colon \left(\bar\delta + 1\right) \xleftrightarrow{\text{o.p.}} \left(X \cup \{\delta\}\right)$:

i) $\pi\left(P^{\bar\delta+1}\left(P^{-1}\left(\bar\delta\right)\right)\right) \simeq P^{\delta+1}\left(\pi\left(P^{-1}\left(\bar\delta\right)\right)\right)$

ii) $\forall k < \omega\ \pi\left(P_k\left(\bar\delta\right)\right) \simeq P_k\left(\delta\right)$

iii) $T\left(\bar\delta\right) \simeq T(\delta)$

iv) $\pi\left(h\left(\bar\delta\right)\right) \simeq h(\delta)$

This is sufficient since these are all new events at this stage. For iii) remember that $\pi\restriction\left(\bar\delta \cap 2\right) = \mathrm{id}$.

Note two easy facts:
Fact 1: If δ is P-closed then $P\left(\langle\delta\rangle\right) = \delta$.
Fact 2: If $\delta = \max\left(P^{-1}(\delta)\right)$ then δ is P-closed.

For i) and ii) it suffices to show that $\pi\left(P^{-1}\left(\bar\delta\right)\right) = P^{-1}(\delta)$.

Case 1: δ is P-closed. Then $\bar{\delta}$ is P-closed. (For $u \in {}^{<\omega}\bar{\delta}$ we have $P(\pi(u)) < \delta$, so $P^{\delta}(\pi(u))\downarrow$. Hence $P^{\bar{\delta}}(u)\downarrow$ with $P(u) < \bar{\delta}$.) By fact 1 we are done.

Case 2: δ is not P-closed. By fact 2 we have $P^{-1}(\delta) \in {}^{<\omega}\delta$. Since $H_{\delta} \subset X$ there is $u \in {}^{<\omega}\bar{\delta}$ with $\pi(u) = P^{-1}(\delta)$. Let $P(u) = \bar{\eta}$. Then $\bar{\delta} \leq \bar{\eta}$, else $\delta > \pi(\bar{\eta}) = \pi\left(P^{\bar{\delta}}(u)\right) = P^{\delta}(\pi(u))\uparrow$. Therefore, $\bar{\delta}$ is not P-closed and so by fact 2 again $P^{-1}(\bar{\delta}) \in {}^{<\omega}\bar{\delta}$. If $\bar{\delta} < \bar{\eta}$ then $\pi\left(P^{-1}(\bar{\delta})\right) <_P \pi(u) = P^{-1}(\delta)$. Thus $P^{\delta}\left(\pi\left(P^{-1}(\bar{\delta})\right)\right)\downarrow$ but $P^{\bar{\delta}}\left(P^{-1}(\bar{\delta})\right)\uparrow$ which contradicts the assumption $\bar{\delta} < \bar{\eta}$. So $\bar{\delta} = \bar{\eta}$ and $\pi\left(P^{-1}(\bar{\delta})\right) = P(u) = P^{-1}(\delta)$.

For iii) and iv) remember that, by viii) of proposition 2, π preserves syntactical notions. So $\bar{\delta}$ is a sentence iff δ is. If not the result is trivial. So assume $\bar{\delta}$ is a sentence.

Case 1: $\bar{\delta}$ is of one of the following forms: $s = t$, $s \in t$, $\neg\, \psi$ or $\psi \vee \chi$ (some terms s, t and formulas ψ, χ) Then iii) is clear by the definition of T. And in iv) both sides are not defined.

Case 2: $\bar{\delta}$ is of the form $(\exists_{\alpha} v_i)\, \psi\,(v_i)$. Then δ is of the form $\left(\exists_{\pi(\alpha)} v_i\right) \pi\,(\psi\,(v_i))$. If $\bar{\delta}$ is true then $T\,(\psi[t]) = 1$ for $\ulcorner t \urcorner = h\left(\bar{\delta}\right)$. So $T\,(\pi(\psi)\,[\pi(t)]) = T(\delta) = 1$ and $\pi\left(h\left(\bar{\delta}\right)\right) \geq h(\delta)$. On the other hand, suppose δ is true. Then $h(\delta) \in H_{\delta} \subset X$. So there is a term t with $\ulcorner t \urcorner \in \bar{\delta}$ and $\pi\,(\ulcorner t \urcorner) = h(\delta)$. Now $T\,(\pi(\psi)\,[\pi(t)]) = T\,(\psi\,[t]) = T\left(\bar{\delta}\right) = 1$ and $\pi\left(h\left(\bar{\delta}\right)\right) \leq h\,(\delta)$.

It follows that $T\left(\bar{\delta}\right) = T(\delta)$ and $\pi\left(h\left(\bar{\delta}\right)\right) = h(\delta)$. $\dashv$

The following definition finally connects the KRS-machine to L by assigning each term of $\mathcal{L}$, i.e., the corresponding ordinal, an element of L.

Definition 16 (Assignment) Define $A\colon \{\ulcorner t \urcorner\,|t \text{ term}\} \to L$ by induction on rank. Note that there are no terms of rank 0 or limit rank. Assume the definition is done for all $\beta \leq \alpha$ and t is a term of rank $\alpha+1$, $t = (S_{\alpha} v_i)\, \varphi\,(v_i)$ say. So φ has rank $\leq \alpha$. Let $\varphi'\left(v_i, v_{k_0}, \ldots, v_{k_{n-1}}\right)$ be a formula of rank $\leq \alpha$ containing no terms and let t_j be terms with $\mathrm{rk}\,(t_j) < \alpha < \mathrm{rk}(t)$, $j < n$, s.t.

$\varphi(v_i) = \varphi'[v_i, t_0, \dots, t_{n-1}]$. So $A(t_j)$ has been defined. Now define

$$A(t) = \left\{x \in L_\alpha \,\middle|\, L_\alpha \models \varphi'[x, A(t_0), \dots, A(t_{n-1})]\right\}.$$

Proposition 3 $L_\alpha = \{A(t) \,|\, t \textit{ is a term of rank } \leq \alpha\}$

Proof This is clear for $\alpha = 0$ and $\lim \alpha$. So assume $\alpha = \beta + 1$: Suppose $x = \{y \in L_\beta \,|\, L_\beta \models \psi[y, t_0, \dots, t_{n-1}]\} \in L_\alpha$, where $\psi\left(v_i, v_{k_0}, \dots, v_{k_{n-1}}\right)$ is a formula of set theory (W.l.o.g. let no variable occur free and bounded.) and $t_j \in L_\beta$, $j < n$. Let φ be the formula of $\mathcal{L}$ obtained from ψ by replacing every occurrence of $\exists$ by $\exists_\beta$ and by the induction hypothesis let s_j, $j < n$, be terms of rank $\leq \beta$ s.t. $A(s_j) = t_j$. Now $s = (S_\beta v_i)\,\varphi(v_i, s_0, \dots, s_{n-1})$ is a term of rank $\beta + 1$ s.t. $A(s) = x$.

Conversely, suppose t is a term of rank $\leq \alpha$. We remain with the case that $t = (S_\beta v_i)\,\varphi(v_i)$ is actually a term of rank α. Let φ' and t_j, $j < n$, be as in the definition. Since $\mathrm{rk}(t_j) < \beta$, $A(t_j) \in L_\beta$, $j < n$. Hence $A(t)$ is definable over L_β. So $A(t) \in L_\alpha$. $\dashv$

Corollary 1 *Let t be a term of rank α. Then $A(t) \in L_\alpha$. Furthermore, $A(l_\alpha) = L_\alpha$, where $l_\alpha = (S_\alpha v_0)(v_0 = v_0)$. So the truth function and the Skolem function are as expected:*

Let $\varphi(x_0, \dots, x_{n-1})$ be a formula of set theory and $\alpha \in \mathrm{On}$. If φ_α is the formula of $\mathcal{L}$ obtained from φ by replacing $\exists$ everywhere by $\exists_\alpha$ and if t_j, $j < n$, are terms of rank $< \alpha$ then

$$T(\varphi_\alpha[t_0, \dots, t_{n-1}]) = 1 \textit{ iff } L_\alpha \models \varphi[A(t_0), \dots, A(t_{n-1})]. \qquad \dashv$$

Definition 17 A machine M is *ZF^--absolute* iff the relation $M_i^\delta(u) \simeq \alpha$ is $\Delta_1^{ZF^-}$ with parameters u, δ, i, and α.

Theorem 5 *The KRS-machine is ZF^--absolute.*

Proof Of course the numbering of the functions is not important. For definiteness, say $M_0 = P$, $M_1 = T$, $M_2 = h$, and $M_{i+3} = P_i$ for all $i < \omega$.

To see that $P^\delta(u) \simeq \alpha$ is $\Delta_1^{ZF^-}$ we first show that $u <_P^\delta v$ is $\Delta_0^{ZF^-}$ with parameters u, v, δ, where $<_P^\delta$ means $<_P \cap ({}^{<\omega}\delta)^2$:

$$\begin{aligned}&\exists\beta\in\delta\,(R\,(u,v,\beta)\wedge\forall\gamma\in\delta\,(\beta\in\gamma\rightarrow R\,(u,v,\beta)\wedge R\,(v,u,\beta))\\&\wedge\neg\, R\,(v,u,\beta))\vee(\forall\beta\in\delta\,(R\,(u,v,\beta)\wedge R\,(v,u,\beta))\wedge u<_{lex} v)\,,\end{aligned}$$

where $R\,(u,v,\beta)$ means $\operatorname{occ}(u,\beta)\leq\operatorname{occ}(v,\beta)$:

$$\exists f\in\mathcal{P}\,(\{i\in\operatorname{dom}v\,|v(i)=\beta\}\times\{i\in\operatorname{dom}u\,|u(i)=\beta\})\,f\ 1{-}1$$

Note the importance of u and v being finite sequences. Now, since ${}^{<\omega}\delta$ is a $\Delta_1^{ZF^-}$-term, $P^\delta(u)\simeq\alpha$ is as required:

$$\exists f\,\big(f\colon\big\{v\in{}^{<\omega}\delta\big|\,v<_P u\big\}\leftrightarrow\alpha\wedge\forall v,w\in{}^{<\omega}\delta\,(v<_P w\rightarrow f(v)<f(w))\big)$$

or

$$\begin{aligned}&\forall f\,\big(\big(f\colon\big\{\,v\in{}^{<\omega}\delta\big|\,v<_P u\big\}\rightarrow\delta\wedge\forall v,w\in{}^{<\omega}\delta\,(v<_P w\rightarrow f(v)<f(w))\\&\wedge\ \operatorname{On}(\operatorname{range}f))\rightarrow\operatorname{range}f=\alpha)\end{aligned}$$

Hence it follows that each $P_i^\delta(u)\simeq\alpha$ is $\Delta_1^{ZF^-}$. T is clear by its recursive definition written out in full. Finally, $h^\delta(\beta)\simeq\alpha$ (we write β for the sequence u of length 1):

$$\begin{aligned}&T(\beta)=1\wedge\exists\gamma\in\delta\,\exists i\in\omega\,\exists u\in{}^{<\omega}\delta\,(\beta=P\,(\langle 4,P\,(\langle 1,\gamma\rangle)\,,i{+}6,5\rangle^\frown u)\\&\wedge\ T\,(\ulcorner f\,(u,i,\alpha)\urcorner)=1\wedge\forall\zeta<\alpha\,(\operatorname{term}(\zeta)\rightarrow T\,(\ulcorner f\,(u,i,\zeta)\urcorner)=0))\,,\end{aligned}$$

where $\operatorname{term}(\zeta)$ is clearly restricted (similar to the part decomposing β above) and $f\,(u,i,\zeta)=\psi\,[t]$ with $\zeta=\ulcorner t\urcorner$ and u representing $\psi\,(v_i)$. $f\,(u,i,\zeta)=v$ can be written as

$$\begin{aligned}&\exists s\colon\operatorname{dom}v\rightarrow(\operatorname{dom}u+1)\,(v(0)=u(0)\wedge s(0)=0\\&\wedge\ \forall j<\operatorname{dom}v\,(j{+}1<\operatorname{dom}v\ \wedge\ s(j)+1<\operatorname{dom}u\rightarrow\\&(u\,(s(j)+1)\neq i{+}6\rightarrow v\,(j{+}1)=u\,(s(j)+1)\wedge s\,(j{+}1)=s(j)+1)\\&\wedge(u\,(s(j)+1)=i{+}6\rightarrow s\,(j+\operatorname{dom}k)=s(j)+1\wedge\forall k<\operatorname{dom}t\\&(v\,(j{+}1{+}k)=t(k)\wedge(k{+}1<\operatorname{dom}t\rightarrow s\,(j{+}1{+}k)=\operatorname{dom}u))))\,.\end{aligned}$$

The idea of this formula is to replace the variable v_i in ψ by the term t

where s is a counter. Note that the first $\exists$ can be replaced by $\forall$ since the rest of the formula uniquely defines s; hence this is a $\Delta_1^{ZF^-}$-formula and we are done. $\dashv$

3 Covering

In this section let M be the KRS-machine and A the assignment from Definition 16.

Remark 2 *$0^\sharp$ exists iff there is a nontrivial elementary embedding of L into L. Furthermore, if α, β are limit ordinals and $\pi\colon L_\alpha \to L_\beta$ is an elementary embedding with $\pi(\gamma) \neq \gamma$ for some $\gamma < \overline{\overline{\alpha}}$ then $0^\sharp$ exists. (see theorem V.4.3 in [1])*

Covering Lemma

This proof of the Covering Lemma is essentially due to Silver and Richardson.

Theorem 6 (Jensen's Covering Lemma) *Assume $0^\sharp$ does not exist and let X be an uncountable set of ordinals. Then there exists a $Y \in L$ s.t. $X \subset Y$ and $\overline{\overline{X}} = \overline{\overline{Y}}$.*

Proof Assume that such a Y does not exist. We show that $0^\sharp$ exists.

Let $\upsilon \in \mathrm{On}$ be least s.t. for some uncountable $X \subset \upsilon$ there is no $Y \in L$ with $X \subset Y$ and $\overline{\overline{X}} = \overline{\overline{Y}}$. Fix such an X. By the minimality of υ the following is easily derived:

i) X is cofinal in υ.

ii) $\aleph_0 < \overline{\overline{X}} < \overline{\overline{\upsilon}}$

iii) $L \models$ "υ is a cardinal"
Assume not and let $\mu = \overline{\overline{\upsilon}}^L$ and $f\colon \mu \leftrightarrow \upsilon$ where $f \in L$. Since $\bar{X} := f^{-1}[X] \subset \mu < \upsilon$ there is a $\bar{Y} \in L$ s.t. $\bar{X} \subset \bar{Y} \subset \mu$ and $\overline{\overline{\bar{X}}} = \overline{\overline{\bar{Y}}}$. Hence $Y := f\left[\bar{Y}\right] \in L$ with $X \subset Y \subset \upsilon$ and $\overline{\overline{X}} = \overline{\overline{Y}}$.

iv) $\neg\, \exists Y \in L \left(X \subset Y \wedge L \models \overline{\overline{Y}} < \upsilon\right)$
Assume $Y \in L$ with $X \subset Y$ and $\mu := \overline{\overline{Y}}^L < \upsilon$. Let $f\colon \mu \leftrightarrow Y$ where

$f \in L$. Since $\bar{X} := f^{-1}[X] \subset \mu < \upsilon$ there is a $\bar{Z} \in L$ s.t. $\bar{X} \subset \bar{Z} \subset \mu$ and $\overline{\overline{\bar{X}}} = \overline{\overline{\bar{Z}}}$. Again $f\left[\bar{Z}\right]$ contradicts the assumption.

The following construction is the main part of this proof.

Let $\pi\colon L_{\bar{\upsilon}} \to L_\upsilon$ be elementary for some $\bar{\upsilon}$. Note that, since υ is a cardinal in L, L_υ is a limit of ZF^--models. For $\eta \geq \bar{\upsilon}$ we define the directed partial ordering $\langle B_\eta, \leq\rangle$:

$$B_\eta = \left\{\langle \delta, F, \beta\rangle \,\middle|\, \beta \leq \delta < \eta \wedge F \subset \delta \wedge \overline{\overline{F}} < \aleph_0 \wedge \beta < \bar{\upsilon}\right\}$$

For $b \in B_\eta$ we write $b = \langle \delta_b, F_b, \beta_b\rangle$. For $b, b' \in B_\eta$: $b \leq b'$ iff $\delta_b \leq \delta_{b'}$, $\beta_b \leq \beta_{b'}$, and $F_b \subset M^{\delta_{b'}}\left[F_{b'} \cup \beta_{b'}\right]$. Note that if $b \leq b'$ then $M^{\delta_b}\left[F_b \cup \beta_b\right] \subset M^{\delta_{b'}}\left[F_{b'} \cup \beta_{b'}\right]$ (i.e., $\gamma_b \leq \gamma_{b'}$, see below).

Now define f_b and γ_b by $f_b\colon \gamma_b \cong M^{\delta_b}\left[F_b \cup \beta_b\right]$. Due to remark 1 we have $\gamma_b = M^{\gamma_b}\left[f_b^{-1}\left(F_b\right) \cup \beta_b\right]$. Furthermore, define $f_{bb'}\colon \gamma_b \to \gamma_{b'}$ $(b \leq b')$ by $f_{bb'} = f_{b'}^{-1} \circ f_b$. Then $f_{bb'}$ is structure-preserving. The following diagram might clear the situation:

$$\begin{array}{ccc} M_b^{\delta}\left[F_b \cup \beta_b\right] & \hookrightarrow & M_{b'}^{\delta'}\left[F_{b'} \cup \beta_{b'}\right] \\ \big\uparrow{\scriptstyle f_b} & & \big\uparrow{\scriptstyle f_{b'}} \\ \gamma_b & \xrightarrow{f_{bb'}} & \gamma_{b'} \end{array}$$

Note that $f_{bb'}$ is the uniquely determined function $f\colon \gamma_b \to \gamma_{b'}$ which is structure-preserving with $f \restriction \beta_b = \mathrm{id} \restriction \beta_b$ and $f\left(f_b^{-1}\left(F_b\right)\right) = f_{b'}^{-1}\left(F_b\right)$. Since M is ZF^--absolute and $L_{\bar{\upsilon}}$ is a limit of ZF^--models, we get that if $\gamma_b, \gamma_{b'} < \bar{\upsilon}$ where $b \leq b'$ then $f_{bb'} \in L_{\bar{\upsilon}}$. Now define the directed system Z_η as $\langle \gamma_b, \in\rangle_{b \in B_\eta}, \langle f_{bb'}\rangle_{b \leq b'}$. It follows that $\pi\left(Z_\eta\right) = \langle \pi\left(\gamma_b\right), \in\rangle_{b \in B_\eta}, \langle \pi\left(f_{bb'}\right)\rangle_{b \leq b'}$ exists iff $\forall b \in B_\eta\ \gamma_b < \bar{\upsilon}$.

From now on, assume that η is a limit ordinal s.t. $\pi\left(Z_\eta\right)$ exists and is well-founded. Let $\langle \eta^*, \in, f_b^*\rangle$ be the direct limit of $\pi\left(Z_\eta\right)$. For $b \leq b'$, $\pi\left(f_{bb'}\right)$ is structure-preserving because of $f_{bb'}$. So by the direct limit property $f_b^*\colon \pi\left(\gamma_b\right) \to \eta^*$ is structure-preserving for each $b \in B_\eta$.

Due to $\lim \eta$ the direct limit of Z_η is $\langle \eta, \in, f_b \rangle_{b \in B_\eta}$. So we can define a function $\pi^* \colon \eta \to \eta^*$ by requiring that $\pi^* \circ f_b = f_b^* \circ \pi$ for all $b \in B_\eta$. To see that π^* is well-defined we must check that the definition is independent of the choice of b: Assume $b \leq b'$, $\alpha \in \gamma_b$, and $\beta \in \gamma_{b'}$ with $f_b(\alpha) = f_{b'}(\beta)$, i.e., $f_{bb'}(\alpha) = \beta$. On the other hand, we have $f_{b'}^* \circ \pi\,(f_{bb'})\,(\pi(\alpha)) = f_b^*\,(\pi(\alpha))$. With $\pi\,(f_{bb'})\,(\pi(\alpha)) = \pi(\beta)$ we infer $f_b^* \circ \pi(\alpha) = f_{b'}^* \circ \pi(\beta)$ as required.

Furthermore, π^* extends $\pi \restriction \bar{\upsilon}$: For any $\beta < \bar{\upsilon}$ we have $b := \langle \beta, \emptyset, \beta \rangle \in B_\eta$, $f_b = \mathrm{id} \restriction \beta$, $f_{bb'} \restriction \beta = \mathrm{id}$, and hence $\pi\,(f_{bb'}) \restriction \pi(\beta) = \mathrm{id}$ for all $b' \geq b$. Then $f_b^* = \mathrm{id} \restriction \pi(\beta)$ and hence $\pi^* \restriction \beta = \pi^* \circ f_b = f_b^* \circ \pi = \pi \restriction \beta$.

Next we show $\pi^* \colon \eta \cong \eta^*$, i.e.,

$$\forall i \in \omega \ \forall u \in {}^{<\omega}\eta \ \pi^*\,(M_i^\eta(u)) \simeq M_i^{\eta^*}(u^*)\,,$$

where $u^* = \pi^*(u)$:

First assume that $y^* := M_i^{\eta^*}(u^*)\downarrow$. Choose $b \in B_\eta$ s.t. $y^* \in \mathrm{range}\, f_b^*$ and $u = f_b\,(u_b)$, where $u_b \in {}^{<\omega}\gamma_b$. Then $u^* = \pi^*(u) = \pi^* \circ f_b\,(u_b) = f_b^* \circ \pi\,(u_b)$. Since f_b^* is structure-preserving, we have $y^* = f_b^* \circ M_i^{\pi(\gamma_b)}\,(\pi\,(u_b))$. Hence $L_\upsilon \models \exists y \in \pi\,(\gamma_b)\ M_i^{\pi(\gamma_b)}\,(\pi\,(u_b)) = y$. By the elementarity of π we infer $L_{\bar{\upsilon}} \models \exists y \in \gamma_b\ M_i^{\gamma_b}\,(u_b) = y$. Now if $y_b = M_i^{\gamma_b}\,(u_b)$ then $f_b\,(y_b) = M_i^\eta(u)\downarrow$ and $f_b^* \circ \pi\,(y_b) = y^*$. Hence $\pi^*\,(M_i^\eta(u)) = \pi^* \circ f_b\,(y_b) = f_b^* \circ \pi\,(y_b) = y^*$ as desired.

Conversely assume $M_i^\eta(u)\downarrow$. Again choose $b \in B_\eta$ s.t. $u = f_b\,(u_b)$ and $M_i^\eta(u) = f_b\,(y_b)$. Since f_b is structure-preserving, we have $y_b = M_i^{\gamma_b}\,(u_b)$. Therefore, using $\pi^*(u) = \pi^* \circ f_b\,(u_b) = f_b^* \circ \pi\,(u_b)$, we get $\pi^*\,(M_i^\eta(u)) = f_b^* \circ \pi\,(y_b) = M_i^{\eta^*}\,(f_b^* \circ \pi\,(u_b)) = M_i^{\eta^*}\,(u^*)$.

Using the following lemma (which will be shown later) we finally prove the existence of $0^\sharp$.

Lemma 1 *There exist $\bar{\upsilon} \in \mathrm{On}$ and $\pi \colon L_{\bar{\upsilon}} \to L_\upsilon$ elementary, s.t.*

i) $X \subset \mathrm{range}\,\pi$

ii) $\pi\restriction\bar{\upsilon}$ *is non-trivial.*

iii) If $\eta \geq \bar{\upsilon}$ *and* $\pi(Z_\eta)$ *exists then* $\pi(Z_\eta)$ *is well-founded.*

Letting $\bar{\upsilon}$ and π be as in the lemma, we assume that $\pi(Z_\eta)$ does not exist for some $\eta \geq \bar{\upsilon}$ (and, therefore, for all $\eta' \geq \eta$). The set $\{\eta \geq \bar{\upsilon} \,|\, \pi(Z_\eta) \text{ exists}\}$ is closed (for $\lim \eta$ we have $B_\eta = \bigcup_{\zeta<\eta} B_\zeta$) and non-empty ($\pi(Z_{\bar{\upsilon}})$ exists). Let η be the largest ordinal s.t. $\pi(Z_\eta)$ exists. We infer the following two properties for η:

i) $\exists F \in [\eta]^{<\omega}\ \exists \beta < \bar{\upsilon}\ \eta = M^\eta[F \cup \beta]$:

   Since $\pi(Z_{\eta+1})$ does not exist, there is a $b \in B_{\eta+1}$ with $\gamma_b \geq \bar{\upsilon}$. Assume $\gamma_b < \eta$; then $b' := \left\langle \gamma_b, f_b^{-1}(F_b), \beta_b \right\rangle \in B_\eta$ and so $\gamma_{b'} < \bar{\upsilon}$. Using $\gamma_b = M^{\gamma_b}\left[f_b^{-1}(F_b) \cup \beta_b\right]$ this implies $\gamma_{b'} = \gamma_b < \bar{\upsilon}$ which is a contradiction. Hence $\eta \leq \gamma_b < \eta + 1$ and, therefore, $F := f_b^{-1}(F_b)$ and $\beta := \beta_b$ are as required.

ii) $\lim \eta$:

   Assume η is of the form $\delta + 1$ and H_δ is from FSP. With F and β as above $\delta = \delta \cap M^\eta[F \cup \beta] \subset M^\delta[((F \cup H_\delta) \cap \delta) \cup \beta]$ holds by the finiteness property. For $b := \langle \delta, (F \cup H_\delta) \cap \delta, \beta \rangle \in B_\eta$ we get $\delta = \gamma_b \geq \bar{\upsilon}$ which contradicts the existence of $\pi(Z_\eta)$.

Finally, let $\langle \eta^*, \in, f_b^* \rangle$ be the direct limit of $\pi(Z_\eta)$ and π^* the extension of π as in the construction above. For F and β as before let $F^* = \pi^*(F)$ and $\beta^* = \pi^*(\beta)$. Then $X \subset \text{range}\,\pi^* = M^{\eta^*}[F^* \cup \beta^*] =: Y$. Hence $Y \in L$ by ZF^--absoluteness and $L \models \left(\overline{\overline{Y}} = \max\left(\aleph_0, \overline{\overline{\beta^*}}\right) < \upsilon\right)$. This contradicts iv) from the beginning of the proof. Hence $\pi(Z_\eta)$ exists for all $\eta \geq \bar{\upsilon}$.

For a cardinal $\eta \geq \bar{\upsilon}$ construct $\pi^*\colon \eta^+ \to (\eta^+)^*$ as before. We extend $\pi^*\restriction\eta$ to an elementary embedding $\hat{\pi}\colon L_\eta \to L_{\eta^*}$ by letting $\hat{\pi}(A(t)) = A(\pi^*(t))$ where t is a term of rank $< \eta$ and $\pi^*\restriction\eta\colon \eta \cong \eta^*$, i.e., $\eta^* = \pi^*(\eta)$:

First note that since η is a cardinal it is P-closed. So $\ulcorner t \urcorner < \eta$ holds for each term with $\operatorname{rk} t < \eta$. Using that $\pi^* \restriction \eta$ is a collapsing function, it follows that $\hat{\pi}$ is well-defined (if, for terms t_1 and t_2, $A(t_1) = A(t_2)$ then we get $T^{\eta}(t_1 = t_2)$, hence $T^{\eta^*}(\pi^*(t_1) = \pi^*(t_2))$, and, finally, $A(\pi^*(t_1)) = A(\pi^*(t_2))$). To see that $\hat{\pi}$ is elementary let $\varphi(v_0, \dots, v_{n-1})$ be a formula of set theory and $x_0, \dots, x_{n-1} \in L_\eta$. For $i < n$ choose terms t_i of $\mathcal{L}$ with $\operatorname{rk} t_i < \eta$ and $A(t_i) = x_i$. Using corollary 1 and the fact that π^* preserves syntactical notions the following chain of equivalences holds:

$$
\begin{aligned}
& L_\eta \models \varphi[x_0, \dots, x_{n-1}] \\
& \text{iff } T^{\eta^+}(\varphi_\eta[t_0, \dots, t_{n-1}]) = 1 \\
& \text{iff } T^{(\eta^+)^*}\big(\varphi_{\pi^*(\eta)}[\pi^*(t_0), \dots, \pi^*(t_{n-1})]\big) = 1 \\
& \text{iff } L_{\eta^*} \models \varphi[\hat{\pi}(x_0), \dots, \hat{\pi}(x_{n-1})]
\end{aligned}
$$

Therefore, ordinals remain ordinals and hence $\hat{\pi}$ extends $\pi^* \restriction \eta$. Finally, $\hat{\pi}$ moves an ordinal since π does. Together with remark 2, this implies the existence of $0^\sharp$. $\dashv$

To finish the proof of the Covering Lemma it remains to show lemma 1. First some considerations.

If $Z = \langle \gamma_b, \in \rangle_{b \in B}, \langle f_{bb'} \rangle_{b \leq b'}$ is a directed system and there is a sequence $\langle w_i \rangle_{i<\omega}$ s.t. there is $\langle b_i \rangle_{i<\omega} \in {}^\omega B$ with $w_i \in \gamma_{b_i}$, $b_i \leq b_j$, and $w_j \in f_{b_i b_j}(w_i)$ for all $i \leq j$, then Z is not well-founded ($f_{b_i}(w_i) = f_{b_j} \circ f_{b_i b_j}(w_i) > f_{b_j}(w_j)$ for $i < j$). We call $\langle w_i \rangle_{i<\omega}$ a *witness for the non-well-foundedness of Z*.

We summarize the main properties of the construction in the main part of the proof by calling a directed system $Z = \langle \gamma_b, \in \rangle_{b \in B}, \langle f_{bb'} \rangle_{b \leq b'}$ for which the following holds an *(υ, β)-construction*: For all $b \in B$ there exist $\beta_b < \gamma_b$ and finite sets $G_b \subset \gamma_b$ s.t.

i) $\gamma_b = M^{\gamma_b}[G_b \cup \beta_b] < \upsilon$,

ii) for $b \leq b'$: $\beta_b \leq \beta_{b'}$, $f_{bb'}: \gamma_b \to \gamma_{b'}$ is structure-preserving, and $f_{bb'} \upharpoonright \beta_b = \mathrm{id} \upharpoonright \beta_b$,

iii) for all $b \in B$ there is a $b' \in B$ s.t. $b \leq b'$ and range $f_{bb'}$ is not cofinal in $\gamma_{b'}$, and

iv) $\beta = \mathrm{lub}\,\{\beta_b | \, b \in B\} \leq \upsilon$.

For an (υ, β)-construction Z and $Y \preceq L_\upsilon$ we write $Z \Subset Y$ iff, for all $b, b' \in B$ with $b \leq b'$, $\gamma_b, \beta_b, G_b, f_{bb'} \in Y$. In this case let $\pi\colon L_{\bar{\upsilon}} \cong Y$ and $\pi^{-1}(Z)$ be the directed system $\langle \pi^{-1}(\gamma_b), \in \rangle_{b \in B}, \langle \pi^{-1}(f_{bb'}) \rangle_{b \leq b'}$. Z is called *Y-well-founded* iff $\pi^{-1}(Z)$ is well-founded. This is true iff there is no witness $\langle w_i \rangle_{i<\omega}$ for the non-well-foundedness of Z where each $w_i \in Y$.

Finally, three preliminary propositions before we prove the lemma.

Proposition 4 *For $Y \preceq L_\upsilon$ and $\beta \leq \upsilon$ let $\pi\colon L_{\bar{\upsilon}} \cong Y$ and $Z \Subset Y$ be an (υ, β)-construction which is Y-well-founded, but not well-founded. Assume $Y \subset Y' \preceq L_\upsilon$, Y' contains a witness for the non-well-foundedness of Z, $Z' \Subset Y$ is an (υ, β')-construction where $\beta \leq \beta' \leq \upsilon$, δ is the direct limit of $\pi^{-1}(Z)$, and $\delta' \geq \delta$ is the direct limit of $\pi^{-1}(Z')$, then Z' is not Y'-well-founded.*

Proof To fix the situation, say $Z = \langle \gamma_b, \in \rangle_{b \in B}, \langle f_{bb'} \rangle_{b \leq b'}$ and $Z' = \langle \gamma_b, \in \rangle_{b \in B'}, \langle f_{bb'} \rangle_{b \leq' b'}$ where B and B' are disjoined. Let $C = B \cup B'$ and for $b \in C$ let $G_b \subset \gamma_b$ and $\beta_b < \gamma_b$ show that Z and Z' are (υ, β)- and (υ, β')-constructions respectively. Furthermore, let $\bar{\gamma}_b = \pi^{-1}(\gamma_b)$, $\bar{G}_b = \pi^{-1}(G_b)$, $\bar{\beta}_b = \pi^{-1}(\beta_b)$, and $\bar{f}_{bb'} = \pi^{-1}(f_{bb'})$ if $b \leq b'$ or $b \leq' b'$. Let $\langle \delta, \in, \bar{f}_b \rangle_{b \in B}$ and $\langle \delta', \in, \bar{f}_b \rangle_{b \in B'}$ be the direct limits of $\pi^{-1}(Z)$ and $\pi^{-1}(Z')$ where $\delta \leq \delta'$ by assumption. Now define $\leq_C$ s.t. $(\leq \cup \leq') \subset \leq_C$ and for $b \in B$ and $b' \in B'$ $b \leq_C b'$ if $\bar{\beta}_b \leq \bar{\beta}_{b'}$ and range $\bar{f}_b \subset$ range $\bar{f}_{b'}$. In this case let $\bar{f}_{bb'} = \bar{f}_{b'}^{-1} \circ \bar{f}_b$.

By $\gamma_b = M^{\gamma_b}[G_b \cup \beta_b]$ we know $\bar{\gamma}_b = M^{\bar{\gamma}_b}\left[\bar{G}_b \cup \bar{\beta}_b\right]$. Clearly $\bar{f}_{bb'}$ is structure-preserving and $\bar{f}_{bb'} \upharpoonright \bar{\beta}_b = \mathrm{id} \upharpoonright \bar{\beta}_b$. As above, each $\bar{f}_b$ is structure-preserving.

Next we show that C is a directed partial ordering. It suffices to show that for all $b \in B$ there is a $b' \in B'$ with $b \leq_C b'$. So assume $b \in B$ and choose a $b_1 \in B'$ with $\beta_b \leq \beta_{b_1}$ which can be done since $\beta \leq \beta'$. Next choose a $b_2 \in B'$ s.t. $b_1 \leq' b_2$ and $\bar{f}_b\left(\bar{G}_b\right) \subset \text{range}\, \bar{f}_{b_2}$ (note that $\bar{f}_b\left(\bar{G}_b\right) \subset \delta$ and $\delta' \geq \delta$ is the direct limit of Z'). Now note that there is an $b_3 \in B$ with $b \leq b_3$ and range f_{bb_3} not cofinal in γ_{b_3}, and hence range $\bar{f}_{bb_3}$ not cofinal in $\bar{\gamma}_{b_3}$. This implies that range $\bar{f}_b$ is bounded by an element of range $\bar{f}_{b_3}$ and hence not cofinal in δ. Therefore, we can choose a $b_4 \geq' b_2$ s.t. $\text{lub}\left(\text{range}\, \bar{f}_b\right) \in \text{range}\, \bar{f}_{b_4}$. Finally, $\text{range}\, \bar{f}_b \subset M^{\text{lub}\left(\text{range}\, \bar{f}_b\right)}\left[\bar{f}_b\left(\bar{G}_b\right) \cup \bar{\beta}_b\right] \subset \text{range}\, \bar{f}_{b_4}$. So b_4 is as desired.

Now it is clear that $\bar{Z}_C = \langle \bar{\gamma}_b, \in \rangle_{b \in C}, \left\langle \bar{f}_{bb'} \right\rangle_{b \leq_C b'}$ is a directed system. Let $Z_C = \pi\left(\bar{Z}_C\right)$. Since B' is cofinal in $\langle C, \leq_C \rangle$ and $\delta' \geq \delta$ we have that Z' and Z_C have the same direct limit. Let $\langle w_i \rangle_{i<\omega} \in Y'$ be a witness for the non-well-foundedness of Z s.t. $w_i \in \gamma_{b_i}$ and $w_j \in f_{b_i b_j}(w_i)$ where $b_i, b_j \in B$ and $i \leq j$. Choose $b_i' \in B'$ with $b_i \leq_C b_i'$ and $\forall j < i\ b_j' \leq' b_i'$ for all $i < \omega$. Finally, let $w_i' = \pi\left(\bar{f}_{b_i b_i'}\right)(w_i)$. $\langle w_i' \rangle_{i<\omega} \in Y'$ is a witness for the non-well-foundedness of Z'. $\dashv$

Proposition 5 *For $Y \preceq L_\upsilon$ there exists a Y' with $Y \subset Y' \preceq L_\upsilon$, $\overline{\overline{Y'}} = \overline{\overline{Y}} + \aleph_0$, and, if $Z \Subset Y$ is a non-well-founded (υ, β)-construction for some $\beta \leq \upsilon$, then Z is not Y'-well-founded.*

Proof Let $\pi\colon L_{\bar{\upsilon}} \cong Y$. We construct Y' by induction. To commence let $Y_0 = Y$ and $\delta_0 = \text{On}$. Our hypothesis is that $Y \subset Y_n \preceq L_\upsilon$, δ_n is defined, $\overline{\overline{Y}} + \aleph_0 = \overline{\overline{Y_n}}$, and if $Z \Subset Y$ is a non-well-founded, Y_n-well-founded (υ, β)-construction for some $\beta \leq \upsilon$ then the direct limit of $\pi^{-1}(Z)$ is in δ_n.

This is true for $n = 0$. For $n \geq 0$ either Y_n is as requested or there is a $Z \Subset Y$ which is a non-well-founded, but Y_n-well-founded (υ, β)-construction for some $\beta \leq \upsilon$. In this case let β be minimal with this property and fix such a Z and a witness for the non-well-foundedness of Z, $\langle w_i \rangle_{i<\omega}$ say. Let Y_{n+1} be the smallest elementary substructure of L_υ with $Y_n \cup \{w_i \,|\, i \in \omega\} \subset Y_{n+1}$

(hence $\overline{\overline{Y_{n+1}}}$ is as required); let δ_{n+1} be the direct limit of $\pi^{-1}(Z)$. By the induction hypothesis we have $\delta_{n+1} \in \delta_n$. We have to check that the induction hypothesis holds for $n+1$: Assume that $Z' \Subset Y$ is a non-well-founded, but Y_{n+1}-well-founded (υ, β')-construction for some $\beta' \leq \upsilon$. By the minimality of β we have $\beta \leq \beta'$ and hence the previous proposition implies that the direct limit of $\pi^{-1}(Z')$ is in δ_{n+1}.

Since the sequence $\langle \delta_n \rangle_{n<\omega}$ is strictly decreasing the construction stops after finitely many steps with the desired Y'. $\dashv$

Proposition 6 *There exists a $Y \preceq L_\upsilon$ with $X \subset Y$ and $\overline{\overline{X}} = \overline{\overline{Y}}$. Furthermore, if $Z \Subset Y$ is a Y-well-founded (υ, β)-construction for some $\beta \leq \upsilon$ then Z is well-founded.*

Proof We construct a tower $\langle Y_\alpha \rangle_{\alpha<\omega_1}$ of L_υ-substructures. The union Y_{ω_1} will be the desired Y.

To commence let Y_0 be the smallest elementary substructure of L_υ containing X as a subset. For $\lim \alpha$ let $Y_\alpha = \bigcup_{\gamma<\alpha} Y_\gamma$. For a given Y_α obtain $Y_{\alpha+1}$ by applying the previous proposition. Thus if $Z \Subset Y_\alpha$ is a non-well-founded (υ, β)-construction for some $\beta \leq \upsilon$ then Z is not $Y_{\alpha+1}$-well-founded.

To see that Y_{ω_1} is as required first note that, since X is uncountable, $\overline{\overline{X}} = \overline{\overline{Y_{\omega_1}}}$. It remains to show that if $Z \Subset Y_{\omega_1}$ is a non-well-founded (υ, β)-construction for some $\beta \leq \upsilon$ then Z is not Y_{ω_1}-well-founded. So assume Z is such a non-well-founded (υ, β)-construction. Using a witness we can choose a countable subsystem Z' of Z s.t. Z' remains a non-well-founded (υ, β')-construction for some $\beta' \leq \upsilon$. Now there is an $\alpha < \omega_1$ with $Z' \Subset Y_\alpha$ and hence Z' is not $Y_{\alpha+1}$-well-founded. Since $Y_{\alpha+1} \subset Y_{\omega_1}$ neither Z' nor Z is Y_{ω_1}-well-founded. $\dashv$

Proof (Lemma 1) Let Y be as defined in the preceding proposition and $\pi\colon L_{\bar{\upsilon}} \cong Y$. Then π is as required. By construction, $X \subset \operatorname{range} \pi$. π moves

an ordinal since $X \subset \operatorname{range}\pi$ is cofinal in υ, but $\overline{\overline{X}} = \overline{\overline{Y}} < \overline{\overline{\upsilon}}$. It remains to show that if $\eta \geq \bar{\upsilon}$ and $\pi(Z_\eta)$ exists then $\pi(Z_\eta)$ is well-founded.

Remember that if $\pi(Z_\eta)$ exists then there is a limit ordinal $\eta' \geq \eta$ s.t. $\pi\left(Z_{\eta'}\right)$ exists. Additionally $\pi\left(Z_{\eta'}\right)$ is a (υ, υ)-construction; to see this use $\lim \eta'$, $\left\langle \beta_b \,\middle|\, b \in B_{\eta'} \right\rangle$ cofinal in $\bar{\upsilon}$, and $\operatorname{range}\pi$ cofinal in υ. Furthermore, $\pi\left(Z_{\eta'}\right)$ is Y-well-founded, since $Z_{\eta'}$ is well-founded. Hence $\pi\left(Z_{\eta'}\right)$ is well-founded by the preceding proposition. Finally, the same is true for any $\pi\left(Z_{\eta''}\right)$ where $\eta'' \leq \eta'$ and hence for $\pi(Z_\eta)$. $\dashv$

This completes the proof of the Covering Lemma.

Strong Covering Lemma

Theorem 7 (Strong Covering Lemma) *Assume* $0^\sharp$ *does not exist and let* $M \in V$ *be a model with universe* $\alpha \subset \mathrm{On}$ *and countable length. Then for any uncountable* $X \subset \alpha$ *there exists* $Y \in L$ *s.t.* $M{\restriction}Y \preceq M$, $X \subset Y \subset \alpha$ *and* $\overline{\overline{X}} = \overline{\overline{Y}}$.

Before we prove the theorem, some well-known definitions.

Definition 18 Let $\alpha \in \mathrm{On}$, κ a cardinal, and $X \subset [\alpha]^\kappa$.

i) X is called *unbounded*[†] iff $\forall y \in [\alpha]^\kappa \; \exists x \in X \; y \subset x$.

ii) X is called *closed*[†] iff for each $\{x_\gamma \,|\, \gamma < \kappa \wedge \forall \beta < \gamma \; x_\beta \subset x_\gamma\} \subset X$ we have $\bigcup\limits_{\gamma<\kappa} x_\gamma \in X$.

iii) X is called *closed-unbounded*[†] (*club*) iff X is closed and unbounded.

iv) X is called *stationary*[†] iff for all club subsets Y of $[\alpha]^\kappa$, X and Y are not disjoined.

[†] actually *in* $[\alpha]^\kappa$ which will be omitted if this is clear from the context

Proof Assume a situation as in the theorem and let $\kappa = \overline{\overline{X}}$. First we show that the following is club:

$$A = \{Y\,|X \subset Y \wedge Y \in [\alpha]^{\kappa} \wedge M{\restriction}Y \preceq M\}$$

A is unbounded by downward Löwenheim-Skolem theorem: Assume $y \in [\alpha]^{\kappa}$, then there is a $Y \supset X \cup y$ with $M{\restriction}Y \preceq M$ and $\overline{\overline{Y}} = \overline{\overline{X \cup y}} = \kappa$ (note that the length of M is countable and hence less than $\aleph_1 \leq \kappa$).

To see that A is closed, note that each $\{Y_\gamma\,|\gamma < \kappa \wedge \forall\beta < \gamma\ Y_\beta \subset Y_\gamma\} \subset A$ is an elementary chain and hence the union is as required.

Next we show that $B_\upsilon^\lambda = \left\{Y \in L \,\middle|\, Y \in [\upsilon]^\lambda\right\}$ is stationary in $[\upsilon]^\lambda$ for all ordinals $\upsilon \geq \lambda$ and uncountable cardinals λ.

Assume not. Let υ be least s.t. for some uncountable cardinal λ there is a club $C \in [\upsilon]^\lambda$ with $B_\upsilon^\lambda \cap C = \emptyset$. Fix λ and C. Similarly to the proof of the Covering Lemma we get:

i) $\aleph_0 < \lambda < \overline{\overline{\upsilon}}$

ii) $L \models$ "υ is a cardinal"
Assume not and let $\mu = \overline{\overline{\upsilon}}^L$ and $f\colon \mu \leftrightarrow \upsilon$ where $f \in L$. Since $\bar{C} := \left\{f^{-1}[c]\,\middle|\, c \in C\right\}$ is club in $[\mu]^\lambda$ with $\mu < \upsilon$ there is a $\bar{c}_0 \in \bar{C} \cap L$. Hence $c_0 := f[\bar{c}_0] \in C \cap L$.

Starting with an elementary $\pi\colon L_{\bar{\upsilon}} \to L_\upsilon$ we construct π^* as above. Again, we need a

Lemma 2 *There exist* $\bar{\upsilon} \in \mathrm{On}$ *and* $\pi\colon L_{\bar{\upsilon}} \to L_\upsilon$ *elementary, s.t.*

i) $\mathrm{range}\,\pi \cap \upsilon \in C$

ii) $\pi{\restriction}\bar{\upsilon}$ *is non-trivial.*

iii) If $\eta \geq \bar{\upsilon}$ *and* $\pi(Z_\eta)$ *exists then* $\pi(Z_\eta)$ *is well-founded.*

With π as in the lemma, we assume that $\pi(Z_\eta)$ does not exist for some $\eta \geq \bar{\upsilon}$. We infer range $\pi^* \in L$ and, therefore, range $\pi^* \cap \upsilon \in C \cap L$ contradicting our assumption. Hence $\pi(Z_\eta)$ exists for all $\eta \geq \bar{\upsilon}$. Again, we can construct $\hat{\pi}\colon L_\eta \to L_{\eta^*}$ for a cardinal η and $0^\sharp$ exists.

Finally, B^κ_α is stationary and $Y \in A \cap B^\kappa_\alpha$ the sought after set. $\dashv$

To prove lemma 2 we need to strengthen some results from the proof of the Covering Lemma.

Proposition 7 *For $Y \preceq L_\upsilon$ and $U \subset \upsilon$ with $\overline{\overline{U}} = \overline{\overline{Y}}$ there exists a Y' with $Y \cup U \subset Y' \preceq L_\upsilon$, $\overline{\overline{Y'}} = \overline{\overline{Y}} + \aleph_0$, and, if $Z \Subset Y$ is a non-well-founded (υ, β)-construction for some $\beta \leq \upsilon$, then Z is not Y'-well-founded.*

Proof In the proof of proposition 5 let Y_0 be the smallest elementary substructure of L_υ with $Y \cup U \subset Y_0$. $\dashv$

Proposition 8 *There exists a $Y \preceq L_\upsilon$ with $Y \cap \upsilon \in C$ and $\overline{\overline{Y}} = \lambda$. Furthermore, if $Z \Subset Y$ is a Y-well-founded (υ, β)-construction for some $\beta \leq \upsilon$ then Z is well-founded.*

Proof We construct a tower $\langle Y_\alpha \rangle_{\alpha < \omega_1}$ of L_υ-substructures. The union Y_{ω_1} will be the desired Y.

To commence choose any element of C, c say, and let Y_0 be the smallest elementary substructure of L_υ containing c as a subset. For $\lim \alpha$ let $Y_\alpha = \bigcup_{\gamma < \alpha} Y_\gamma$. For a given Y_α obtain $Y_{\alpha+1}$ by applying the previous proposition to Y_α and $c_\alpha \in C$ with $Y_\alpha \cap \upsilon \subset c_\alpha$. Thus if $Z \Subset Y_\alpha$ is a non-well-founded (υ, β)-construction for some $\beta \leq \upsilon$ then Z is not $Y_{\alpha+1}$-well-founded.

To see that Y_{ω_1} is as required first note that, since λ is uncountable, $\overline{\overline{Y_{\omega_1}}} = \lambda$. The proof, that if $Z \Subset Y_{\omega_1}$ is a non-well-founded (υ, β)-construction for some $\beta \leq \upsilon$ then Z is not Y_{ω_1}-well-founded, is the same as in proposition 6. Finally, $Y \cap \upsilon = \bigcup_{\alpha < \omega_1} c_\alpha \in C$ since C is closed. $\dashv$

Proof (Lemma 2) Let Y be as given by the preceding proposition and $\pi\colon L_{\bar{\upsilon}} \cong Y$. Then π is as required. By construction, $\operatorname{range}\pi \cap \upsilon \in C$. $\pi \restriction \upsilon$ moves an ordinal since $\operatorname{range}\pi \cap \upsilon \in C \setminus L$ by assumption. The proof, that if $\eta \geq \bar{\upsilon}$ and $\pi\,(Z_\eta)$ exists then $\pi\,(Z_\eta)$ is well-founded, is as in lemma 1 except for $\pi\left(Z_{\eta'}\right)$ being a (υ, β)-construction for some $\beta \leq \upsilon$ (we lose the cofinality of $\operatorname{range}\pi$ in υ). $\dashv$

This completes the proof of the Strong Covering Lemma.

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